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Multipliers of dynamical systems

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Queen's University Belfast

Doctoral Thesis

Multipliers of dynamical systems

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A thesis submitted for the degree of

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Abstract

Herz–Schur multipliers of a locally compact group have a well developed theory coming from a large literature; they have proved very useful in the study of the reduced C^*-algebra of a locally compact group. There is also a rich connection to Schur multipliers, which have been studied since the early twentieth century, and have a large number of applications.

We develop a theory of Herz–Schur multipliers of a C^*-dynamical system, extending the classical Herz–Schur multipliers, making Herz–Schur multiplier techniques available to study a much larger class of C^*-algebras. Furthermore, we will also introduce and study generalised Schur multipliers, and derive links between these two notions which extend the classical results describing Herz–Schur multipliers in terms of Schur multipliers. This theory will be developed in as much generality as possible, with reference to the classical motivation.

After introducing all the necessary concepts we begin the investigation by defining generalised Schur multipliers. The main result is a dilation type characterisation of these multipliers; we also show how such multipliers can be represented using Hilbert C^*-modules. Next we introduce and study generalised Herz–Schur multipliers, first extending a classical result involving the representation theory of SU(2), before studying how such functions are related to our generalised Schur multipliers. We give a characterisation of generalised Herz–Schur multipliers as a certain class of the generalised Schur multipliers, and obtain a description of precisely which Schur multipliers belong to this class. Finally, we consider some ways in which the generalised multipliers can arise; firstly, from the classical multipliers which provide our motivation, secondly, from the Haagerup tensor product of a C^*-algebra with itself, and finally from positivity considerations. We show that our theory behaves well with respect to positivity and give conditions under which our multipliers are automatically positive in a natural sense.

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Contents

Introduction

The entrywise product of matrices representing elements of $\mathcal{B}(\ell^2)$ was systematically studied by Schur [52]; those functions which behave well with respect to this operation now bear his name: a bounded function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ is called a *Schur multiplier* if $(\varphi(i,j)a_{i,j})_{i,j\in\mathbb{N}}$ is an element of $\mathcal{B}(\ell^2)$ whenever $(a_{i,j})_{i,j\in\mathbb{N}}$ is such. The significance of such functions was recognised by Grothendieck [25] (see also Pisier [46, Chapter 5]), who gave a characterisation of Schur multipliers and used them to formulate his famous inequality.

Peller [45] and Haagerup [27] have shown that it is fruitful to study Schur multipliers defined on the product of two measure spaces; they obtained results which contain Grothendieck's characterisation as the special case of counting measure. Peller's work was motivated by the theory of *Double Operator Integrals* (see also Hiai and Kosaki [31]), while Haagerup was interested in the decomposition of completely bounded maps; further evidence of the diverse applications of Schur multiplier techniques can be seen in the work of Katavalos and Paulsen [36], where they use measurable Schur multipliers to study the class of normal bimodule idempotents.

Parallel to the theory of Schur multipliers is the theory of *Herz–Schur multipliers*[1] of a locally compact group. Herz–Schur multipliers originated in work of Herz [30]; Herz viewed this space as a generalisation of the Fourier–Stieltjes transform. Bożejko and Fendler [10], using unpublished work of Gilbert (see also Jolissaint [33]), showed that the Herz–Schur multipliers on G are precisely the completely bounded multipliers of the Fourier algebra of G studied by De Cannière and Haagerup [17]; that is, those

[1] I was informed by A. Derighetti that one should really say Herz–*Varopoulos*–Schur multipliers, due to the influence of Varopoulos's work [58].

scalar-valued functions on the group for which pointwise multiplication is a completely bounded map on the Fourier algebra of the group.

Haagerup and his collaborators [15, 26, 28, 38] have used Herz–Schur multipliers to study approximation properties of C^*-algebras and von Neumann algebras; they use the fact that a Herz–Schur multiplier on a group gives rise to a completely bounded map on the associated reduced group C^*-algebra (and group von Neumann algebra), and show that properties of this completely bounded map reflect properties of the Herz–Schur multiplier. In fact this idea was used by Lance [39, Section 4] in his proof that a discrete group is amenable if and only if its reduced group C^*-algebra is nuclear. A notable success of this programme is Haagerup's proof that the reduced C^*-algebra of the free group $\mathbb{F}_2$ is not a nuclear C^*-algebra but does have Grothendieck's metric approximation property [26].

The work of Bożejko and Fendler [10] also shows that Herz–Schur multipliers of a locally compact group G can be identified with the Schur multipliers on $G \times G$ of Toeplitz type: those scalar-valued functions on $G \times G$ of the form $N(u)(s,t) = u(ts^{-1})$ $(s, t \in G)$, where $u : G \to \mathbb{C}$. A concise account is given by Todorov [56], where invariant Schur multipliers are defined and it is shown that they are precisely those in the image of the Herz–Schur multipliers under the map N.

The large number of applications of Schur and Herz–Schur multipliers described above, as well as their intrinsic interest, has motivated a number of authors to develop more general versions of these multipliers. In keeping with the modern taste for noncommutative, or *quantised*, methods in Functional Analysis [22] Kissin and Shulman [37] have introduced *operator multipliers*, which provide a non-commutative framework generalising that of the classical Schur multipliers discussed so far. Motivated by an idea from the development of Double Operator Integrals Shulman, Todorov, and Turowska [53] studied *closable multipliers* and *local multipliers*, which allow unbounded generalisations of Schur multipliers. In a different direction we have Renault's work on groupoids [49]. Renault introduces the Fourier algebra of a groupoid and defines the multipliers and completely bounded multipliers of this space; this work contains both Schur multipliers and Herz–Schur multipliers as special cases. The applications of Herz–Schur multipliers to the study of approximation properties has led several authors to introduce multipliers acting on crossed products. For example, vector-valued positive-definite functions on a W^*-dynamical system play an important role in work of Anantharaman-Delaroche [1] on amenable actions. More recently Dong and Ruan [20] introduced a version of the Haagerup property for C^*-dynamical systems; motivated by the work of Haagerup above they define a multiplier to be a function from a group to a C^*-algebra, and to each multiplier they associate a map which acts on the reduced crossed product formed from the action of this group on the C^*-algebra. These multipliers are then used to formulate their Hilbert module version of the Haagerup property. Finally, recent work of Bédos and Conti [4, 5] uses multipliers, defined analogously to Herz–Schur multipliers, to study properties of crossed products. It appears that their definitions are the closest to those given here, though Bédos and Conti consider only functions on discrete groups. The connections between the work presented here and some of the ideas mentioned in this paragraph are explored in Section 3.3.

In this thesis we provide another generalisation of Schur and Herz–Schur multipliers. The work cited above of Anantharaman-Delaroche, Dong and Ruan, and Bédos and Conti, suggests that it would be useful to have a general theory of Herz–Schur multipliers of a C^*-dynamical system, which act on the associated reduced crossed product, extending the classical theory of Herz–Schur multipliers of groups, which act on the associated reduced group C^*-algebra (or group von Neumann algebra). Such a theory would provide the tools for a study of C^*-dynamical systems and crossed products similar to the study of groups and group C^*-algebras conducted by Haagerup and his collaborators. The results mentioned above on classical Herz–Schur multipliers suggest that one should also search for a parallel theory of generalised Schur multipliers, and a suitable notion of 'Toeplitz type', such that our generalised Herz–Schur multipliers are precisely the generalised Schur multipliers of Toeplitz type. Moreover, we aim to develop this theory without having to restrict to discrete spaces and counting measure. Achieving these goals is the main focus of this thesis.

Let us now outline the content of each chapter. Chapter 1 contains an overview of the background material which will be used in later chapters. I aim to give a more detailed overview of the classical theory of Schur and Herz–Schur multipliers than is strictly necessary as several of the calculations directly influence later chapters. Chapter 1 also includes an account of vector-valued integration, which will be essential to the results presented, in order to keep track of the technical details which appear.

Chapter 2 is concerned with a generalisation of Schur multipliers to functions which take values acting on a C^*-algebra; first a class of integral kernel operators with C^*-algebra-valued kernels is developed, motivated by the role of Hilbert–Schmidt operators in the definition of classical measurable Schur multipliers, before we introduce Schur multipliers which are defined by their action on the integral kernel operators. The main result is Theorem 2.9, which generalises the classical characterisations of Schur multipliers given by Grothendieck and Peller. Theorem 2.11 shows how our Schur multipliers can be represented using Hilbert C^*-modules.

In Chapter 3 Herz–Schur multipliers of a C^*-dynamical system are introduced and studied. For technical reasons we are forced to consider two similar classes; the first section contains a characterisation of one of these classes which generalises a result of De Cannière and Haagerup [17, Theorem 1.6]. The goal of the second section is to obtain a Transference Theorem, relating Herz–Schur multipliers of a C^*-dynamical system to the Schur multipliers introduced in Chapter 2. This is done in two stages: first a map $\mathcal{N}$, which takes the role of N in the classical transference results, is introduced and Theorem 3.11 shows that $\mathcal{N}$ carries Herz–Schur multipliers to Schur multipliers; secondly we identify the image of the map $\mathcal{N}$ in Theorem 3.20.

Chapter 4 describes classes of Schur and Herz–Schur multipliers which arise in different ways. In the first section we show how classical Schur and Herz–Schur multipliers give rise to the multipliers introduced in Chapters 2 and 3. The second section begins with a description of the Haagerup tensor product; we then show how functions into the Haagerup tensor product naturally give rise to a Schur multiplier as defined in Chapter 2. Finally, we investigate positivity for Schur and Herz–Schur multipliers; this section is motivated by the large number of applications of positive Herz–Schur multipliers in the literature, such as the work of Lance [39, Section 4] and Haagerup [26]. The section begins by considering positivity for measurable Schur multipliers; we obtain a dilation characterisation for completely positive Schur multipliers in Theorem 4.6, similar to the completely bounded version in Chapter 2, and a completely positive version of the Transference Theorem in Proposition 4.7. Specialising to the case of counting measure we give further characterisations of positive multipliers.

The work presented here in Chapters 2, 3, and 4 is available in preprint form [42].

Chapter 1

Preliminaries

In this chapter we lay out the main definitions and results to be used, aiming to fix notation, provide references, and formulate results in a way that generalise naturally to the operator-valued cases we aim to study.

We follow the usual notation for number systems: $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ the complex numbers, with $\overline{\lambda}$ denoting the complex conjugate of $\lambda \in \mathbb{C}$. If X and Y are non-empty sets then $X \times Y$ denotes their Cartesian product. All vector spaces will be over the complex numbers; I will simply write *map* in place of linear map. In the sections below we define a number of structures that will be considered in this thesis. An equality with the symbol $:=$ indicates that the left side is being defined.

Let us first fix some notation for functions on a topological space X. Let f be a function from X to a vector space. The *support of* f is defined by

$$\operatorname{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}.$$

The collection of all continuous functions $f : X \to \mathbb{C}$ such that $\operatorname{supp} f$ is compact will be denoted by $C_c(X)$. The characteristic function of a set $E \subseteq X$ will be written χ_E.

We will mainly be considering topological spaces satisfying the following property.

Definition 1.1. A topological space is called *locally compact* if every point has a neighbourhood basis consisting of compact sets.

I will often assume the following extra properties on a topological space.

Definition 1.2. A topological space is called *separable* if it has a countable dense subset, and *second-countable* if its topology has a countable basis.

The assumption of second-countability will sometimes be used in the following form, given by Cohn [13, Proposition 7.1.6].

Proposition 1.3. *If a topological space X is second-countable and locally compact then it is* σ-compact*; that is, there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets (which may be chosen to be an increasing sequence) such that $X = \cup_{n\in\mathbb{N}} K_n$.* □

1.1 Banach and operator algebras

We begin with the definition of a Banach space.

Definition 1.4. A *norm* on a vector space E is a function $\|\cdot\| : E \to [0, \infty)$ satisfying the conditions:

i. $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in E,\ \lambda \in \mathbb{C}$;

ii. $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in E$;

iii. $\|x\| = 0$ if and only if $x = 0$.

A function satisfying conditions (i) and (ii) is called a *seminorm.* We say that E is a *Banach space* if E is complete in the metric induced by the norm: $d(x, y) := \|x - y\|$ $(x, y \in E)$. The *dual space* E^* of the normed space E is the space of all bounded, linear functionals on E.

If E is a Banach space and ϕ belongs to the dual space E^* then it will sometimes be convenient to write $\langle \phi, x\rangle$ for $\phi(x)$ $(x \in E)$.

We will mainly be studying Banach spaces which are also algebras in a way compatible with the norm.

Definition 1.5. A *Banach algebra* is an algebra E endowed with a norm $\|\cdot\|$, with respect to which E is a Banach space, and satisfying the additional condition:

iv. $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in E$.

Definition 1.6. A *Banach $*$-algebra* is a Banach algebra E endowed with an *involution*, which is a function $* : E \to E$ satisfying the conditions:

i. $(x^*)^* = x$ for all $x \in E$;

ii. $(xy)^* = y^* x^*$ for all $x, y \in E$;

iii. $(\lambda x + y)^* = \overline{\lambda} x^* + y^*$ for all $\lambda \in \mathbb{C},\ x, y \in E$;

such that $\|x\| = \|x^*\|$ for all $x \in E$. I have written x^* in place of $*(x)$ as is customary.

Banach spaces with an inner product structure will be very important.

Definition 1.7. Let E be a vector space. An *inner product* on E is a function

$$\langle \cdot, \cdot \rangle : E \times E \to \mathbb{C}$$

satisfying the conditions:

i. $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$ for all $\lambda, \mu \in \mathbb{C},\ x, y, z \in E$;

ii. $\overline{\langle x, y \rangle} = \langle y, x \rangle$ for all $x, y \in E$;

iii. $\langle x, x \rangle \geq 0$ for all $x \in X$;

iv. $\langle x, x \rangle = 0$ if and only if $x = 0$.

A *Hilbert space* is a vector space E, with an inner product, such that E is a Banach space with respect to the *inner product norm*

$$\|x\| := \langle x, x \rangle^{\frac{1}{2}}, \quad x \in E. \tag{1.1}$$

I will usually use capital calligraphic letters $\mathcal{H}, \mathcal{L}, \ldots$ to denote Hilbert spaces. We will often work with separable Hilbert spaces; note that separability of a Hilbert space $\mathcal{H}$ is equivalent to the existence of a countable orthonormal basis for $\mathcal{H}$ [34, Remark 2.2.14]. The identity operator on a Hilbert space $\mathcal{H}$ will be written $I_{\mathcal{H}}$.

The elementary properties of inner products, Hilbert spaces, Banach spaces, and Banach algebras may be found in Conway [14] for example.

1.1.1 Algebras of operators

Our main focus will be C^*-algebras. The material in this section is based on Dixmier [18] and Davidson [16, Chapter 1].

Definition 1.8. A *C^*-algebra* is a Banach $*$-algebra A with the additional property that

$$\|x^*x\| = \|x\|^2, \quad x \in A.$$

The C^*-algebra A will be called *unital* if there is an element $1_A \in A$ which is an identity for multiplication.

Example 1.9. Let $\mathcal{H}$ be a Hilbert space. We write $\mathcal{B}(\mathcal{H})$ for the $*$-algebra of all bounded linear operators on $\mathcal{H}$, with multiplication given by composition and the usual adjoint operation. This algebra is a C^*-algebra under the operator norm

$$\|T\| := \sup\{\|T\xi\| : \xi \in \mathcal{H},\ \|\xi\| \leq 1\}.$$

When working with C^*-algebras their representations are an important tool.

Definition 1.10. Let A be a C^*-algebra and $\mathcal{H}$ a Hilbert space. A *representation* of A on $\mathcal{H}$ is a $*$-homomorphism from A to $\mathcal{B}(\mathcal{H})$; if ρ is a representation of a C^*-algebra we usually denote the associated Hilbert space by $\mathcal{H}_\rho$, and write "$(\rho, \mathcal{H}_\rho)$ is a representation".

A representation $(\rho, \mathcal{H}_\rho)$ of a C^*-algebra A is called *faithful* if ρ is injective, *irreducible* if $\rho(A)$ has no proper closed invariant subspaces, *non-degenerate* if $\rho(A)\mathcal{H}_\rho$ is dense in $\mathcal{H}_\rho$

(equivalently: for every non-zero element $\xi \in \mathcal{H}_\rho$ there is $a \in A$ such that $\rho(a)\xi \neq 0$), and *cyclic* if there exists a vector $\xi \in \mathcal{H}_\rho$, called a *cyclic vector*, such that $\rho(A)\xi$ is dense in $\mathcal{H}_\rho$.

The same terminology will be used for representations of Banach $*$-algebras.

The order structure on a C^*-algebra will be used in Subsection 1.1.2. Further results on the positive elements of a C^*-algebra are given by Dixmier [18, Section 1.6].

Definition 1.11. Let A be a C^*-algebra. An element $a \in A$ is called *hermitian* if $a = a^*$. A hermitian element $a \in A$ is called *positive* if there exists $x \in A$ such that $a = x^*x$; in this case we write $a \geq 0$. The collection of positive elements of A will be denoted by A^+.

A linear functional ϕ on A is called *positive* if $\phi(a) \geq 0$ for all $a \in A^+$. A *state* on A is a positive linear functional of norm 1.

Note that a positive linear functional on a C^*-algebra is automatically continuous [16, Lemma I.9.5].

The following result, called the *GNS Theorem* after Gelfand, Naimark, and Segal, relates positive linear functionals to representations. The proof, given by Davidson [16, Theorem I.9.6], is called the *GNS construction.*

Theorem 1.12. *Let A be a C^*-algebra and ϕ a positive linear functional on A. Then there exists a representation $\tilde{\phi}$ of A on a Hilbert space $\mathcal{H}_\phi$, and a vector $\xi_\phi \in \mathcal{H}_\phi$ which is cyclic for $\tilde{\phi}$, such that*

$$\phi(a) = \left\langle \tilde{\phi}(a)\xi_\phi, \xi_\phi \right\rangle, \quad a \in A. \qquad \square$$

Davidson [16, Lemma I.9.10] uses the Hahn–Banach Theorem to ensure there exists a supply of states, and therefore of representations, for any C^*-algebra.

We now introduce unitisations and the multiplier algebra of a C^*-algebra; see Pedersen [44, Section 3.12], Raeburn–Williams [48, Section 2.3], and Davidson [16, Proposition I.1.3] for further details.

Definition 1.13. Suppose that A is a C^*-algebra. A *unitisation of* A is a unital C^*-algebra B, and an injective $*$-homomorphism $i : A \to B$, such that $i(A)$ is an essential ideal of B (*i.e.* every closed ideal of B has non-trivial intersection with $i(A)$). Let $A^{\#}$ denote the unitisation $(A \oplus \mathbb{C}, i)$, where i is the inclusion map and the C^*-algebra structure on $A \oplus \mathbb{C}$ is given by

$$\begin{aligned}(a, \lambda)(b, \mu) &:= (ab + \lambda b + \mu a, \lambda\mu)\\ (a, \lambda)^* &:= (a^*, \overline{\lambda})\\ \|(a, \lambda)\| &:= \sup_{\|b\| \leq 1} \|ab + \lambda b\|,\end{aligned}$$

for all $a, b \in A,\ \lambda, \mu \in \mathbb{C}$. The maximal unitisation of A, denoted by $\mathrm{M}(A)$, is called the *multiplier algebra of* A.

For a formal statement of the meaning of 'maximal' here see the universal property given by Raeburn–Williams [48, Definition 2.46]. The references [44, Section 3.12] and [48, Section 2.3] show that the multiplier algebra can be realised in various ways; for example, Pedersen [44, page 78], shows that for any faithful, non-degenerate, representation of A the multiplier algebra $\mathrm{M}(A)$ is contained in A''.

Now we define two further topologies on the algebra of operators on a Hilbert space. See Davidson [16, Section I.6] for further discussion.

Definition 1.14. Let $\mathcal{H}$ be a Hilbert space. The *weak operator topology* on $\mathcal{B}(\mathcal{H})$ is defined to be the weakest topology such that the sets

$$U(T, \xi, \eta) := \{A \in \mathcal{B}(\mathcal{H}) : \big|\langle (T - A)\xi, \eta\rangle\big| < 1\}, \quad T \in \mathcal{B}(\mathcal{H}),\ \xi, \eta \in \mathcal{H},$$

are open. A net $(T_\gamma)_{\gamma\in\Gamma} \subseteq \mathcal{B}(\mathcal{H})$ converges to $T \in \mathcal{B}(\mathcal{H})$ in the weak operator topology if and only if

$$\langle T_\gamma \xi, \eta\rangle \xrightarrow{\gamma} \langle T\xi, \eta\rangle$$

for all $\xi, \eta \in \mathcal{H}$.

The *strong operator topology* on $\mathcal{B}(\mathcal{H})$ is defined to be the weakest topology such that the sets

$$V(T,\xi) := \{A \in \mathcal{B}(\mathcal{H}) : \|(T-A)\xi\| < 1\}, \quad T \in \mathcal{B}(\mathcal{H}),\ \xi \in \mathcal{H},$$

are open. A net $(T_\gamma)_{\gamma\in\Gamma} \subseteq \mathcal{B}(\mathcal{H})$ converges to $T \in \mathcal{B}(\mathcal{H})$ in the strong operator topology if and only if

$$T_\gamma \xi \xrightarrow{\gamma} T\xi$$

for all $\xi \in \mathcal{H}$.

The *Double Commutant Theorem* of von Neumann (*e.g.* Davidson [16, Theorem I.7.1]) describes the weak operator topology closure of a C^*-subalgebra A of $\mathcal{B}(\mathcal{H})$ in terms of its *commutant* A', defined by

$$A' := \{T \in \mathcal{B}(\mathcal{H}) : ST = TS \text{ for all } S \in A\}.$$

Definition 1.15. Let $\mathcal{H}$ be a Hilbert space. A *von Neumann algebra on* $\mathcal{H}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ which contains the identity operator and is closed in the weak operator topology.

1.1.2 Operator spaces and operator systems

The theory of operator spaces will be essential. Here we recall the main definitions and results, and refer to Effros–Ruan [22] for further details. Let E be a vector space and $n \in \mathbb{N}$; I will write $M_n(E)$ for the space of all $n \times n$ matrices with entries in E, which is a ($*$-)algebra under the natural operations if E is so. When $E = \mathbb{C}$ I write M_n in place of $M_n(\mathbb{C})$ and $M_{m,n}$ for the space of $m \times n$ matrices with entries in $\mathbb{C}$.

Definition 1.16. Let V be a vector space. We say V is an *abstract operator space* if there is a norm $\|\cdot\|_n$ on $M_n(V)$ for each $n \in \mathbb{N}$ satisfying the conditions:

i. $\|v \oplus w\|_{m+n} = \max\{\|v\|_m, \|w\|_n\}$;

ii. $\|\alpha v \beta\|_n \leq \|\alpha\|\|v\|_m\|\beta\|$;

for all $v \in M_m(V)$, $w \in M_n(V)$, $\alpha \in M_{n,m}$, $\beta \in M_{m,n}$, where we use the notation $v \oplus w$ for the matrix

$$\begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \in M_{m+n}(V).$$

This norm structure will be called a *matrix norm* on V.

A *concrete operator space* on a Hilbert space $\mathcal{H}$ is a linear subspace of $\mathcal{B}(\mathcal{H})$. If W is such a space then the matrix norms satisfying conditions (i) and (ii) are given by the inclusions

$$M_n(W) \subseteq M_n\big(\mathcal{B}(\mathcal{H})\big) = \mathcal{B}(\mathcal{H}^n), \tag{1.2}$$

which hold for each $n \in \mathbb{N}$.

I will use simply *operator space* throughout when it is not necessary to specify a Hilbert space being acted on.

Naturally, we are interested in maps which respect the operator space structure.

Definition 1.17. Let V and W be operator spaces and $\phi : V \to W$ a linear map. For each $n \in \mathbb{N}$ define a linear map

$$\phi^{(n)} : M_n(V) \to M_n(W);\ \phi^{(n)}(a_{i,j})_{i,j=1}^n := \big(\phi(a_{i,j})\big)_{i,j=1}^n, \quad (a_{i,j})_{i,j=1}^n \in M_n(V).$$

We say ϕ is *completely bounded* if

$$\|\phi\|_{\mathrm{cb}} := \sup_{n\in\mathbb{N}} \|\phi^{(n)}\|$$

is finite. We let $\mathcal{CB}(V,W)$ denote the space of all completely bounded maps from V to W, and endow it with the norm $\|\cdot\|_{\mathrm{cb}}$; $\mathcal{CB}(V,V)$ will be written $\mathcal{CB}(V)$.

We also say that ϕ is a *complete isometry* if each $\phi^{(n)}$ is an isometry, and a *complete isomorphism* if ϕ is a linear bijection such that both $\|\phi\|_{\mathrm{cb}}$ and $\|\phi^{-1}\|_{\mathrm{cb}}$ are finite.

Completely bounded maps are characterised by the following result, which I will refer to as the *Haagerup–Paulsen–Wittstock Theorem.* A proof is given by Paulsen [43, Theorem 8.4] and Brown–Ozawa [11, Theorem B.7].

Theorem 1.18. *Let A be a C^*-algebra, $X \subseteq A$ an operator space, and $\phi : X \to \mathcal{B}(\mathcal{H})$. The following are equivalent:*

i. ϕ is completely bounded;

ii. there exist a representation $(\rho, \mathcal{H}_\rho)$ of A, and bounded operators $V, W : \mathcal{H} \to \mathcal{H}_\rho$, with $\|\phi\|_{\mathrm{cb}} = \|V\|\|W\|$, such that

$$\phi(a) = W^* \rho(a) V, \quad a \in X. \qquad \square$$

A full discussion of the following examples can be found in Effros–Ruan [22].

Examples 1.19. (i) Any C^*-algebra is an operator space. Indeed, if A is a C^*-algebra then choose a faithful representation $(\rho, \mathcal{H}_\rho)$ of A on a Hilbert space. The identification

$$M_n\big(\rho(A)\big) \subseteq M_n\big(\mathcal{B}(\mathcal{H}_\rho)\big) = \mathcal{B}(\mathcal{H}_\rho^n)$$

provides a matrix norm structure on $\rho(A)$, which can be shown to be independent of the choice of faithful representation ρ [22, page 21]. This gives an operator space structure on A.

(ii) More generally, if $\mathcal{H}$ and $\mathcal{L}$ are Hilbert spaces then $\mathcal{B}(\mathcal{H}, \mathcal{L})$, the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{L}$, is an operator space, with the matrix norm given by the identification

$$M_n\big(\mathcal{B}(\mathcal{H}, \mathcal{L})\big) \cong \mathcal{B}(\mathcal{H}^n, \mathcal{L}^n)$$

for each $n \in \mathbb{N}$. Alternatively, one can identify $\mathcal{B}(\mathcal{H}, \mathcal{L})$ with the subspace of matrices of the form

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{L} \oplus \mathcal{H}), \quad a \in \mathcal{B}(\mathcal{H}, \mathcal{L}),$$

and use the matrix norm on $\mathcal{B}(\mathcal{L} \oplus \mathcal{H})$ given by (1.2).

We write $\mathcal{K}(\mathcal{H}, \mathcal{L})$ for the space of all compact operators from $\mathcal{H}$ to $\mathcal{L}$, that is, the norm closure of the space of finite rank operators from $\mathcal{H}$ to $\mathcal{L}$, which inherits an operator space structure from $\mathcal{B}(\mathcal{H}, \mathcal{L})$; the notation $\mathcal{K}(\mathcal{H})$ will be used for the C^*-algebra of all compact operators on $\mathcal{H}$.

(iii) If V is an operator space then the dual space V^* is an operator space. To see this we use the identification of V^* with $\mathcal{CB}(V, \mathbb{C})$ [22, Corollary 2.2.3] and the identification

$$M_n(\mathcal{CB}(V, \mathbb{C})) = \mathcal{CB}(V, M_n),$$

which gives a matrix norm structure on V^*. See Effros–Ruan [22, Section 3.2].

The following representation theorem for operator spaces shows that every operator space can be represented as a concrete operator space. The proof is given by Effros–Ruan [22, Theorem 2.3.5].

Theorem 1.20. *Let V be an abstract operator space. Then there is a Hilbert space $\mathcal{H}_\Phi$, a concrete operator space $W \subseteq \mathcal{B}(\mathcal{H}_\Phi)$, and a complete isometry Φ of V onto W.* $\square$

We will also make use of the parallel notion of positivity at matrix levels. For this we must introduce operator systems, following Brown–Ozawa [11, Section 1.5] and Effros–Ruan [22, Chapter 5]. Recall from Definition 1.11 that any C^*-algebra A possesses an order structure; if $n \in \mathbb{N}$ then $M_n(A)$ is also a C^*-algebra, and therefore also carries an order structure.

Definition 1.21. A (concrete) *operator system* is a norm-closed, unital, self-adjoint, subspace of a unital C^*-algebra. If V is an operator system contained in a C^*-algebra A then, for each $n \in \mathbb{N}$, $M_n(V)$ inherits an adjoint operation, order structure, and matrix norms, from $M_n(A)$. The positive elements of $M_n(V)$ are denoted by $M_n(V)^+$.

Now we define maps which respect this matrix order structure.

Definition 1.22. Let V and W be operator systems (or C^*-algebras) and $\phi : V \to W$ a linear map. We say ϕ is *completely positive* if $\phi^{(n)}$ is a positive map for each $n \in \mathbb{N}$; that is, if $x \in M_n(V)^+$ then $\phi^{(n)}(x) \in M_n(W)^+$.

Note that completely positive maps are automatically completely bounded [22, Lemma 5.1.1]. Completely positive maps are characterised by *Stinespring's Theorem*, given below. We refer to Effros–Ruan [22, Theorem 5.2.1] for the proof; see also Brown–Ozawa [11, Remark 1.5.4 and Proposition 2.2.1] for a discussion of non-unital technicalities.

Theorem 1.23. *Let A be a C^*-algebra, and let $\phi : A \to \mathcal{B}(\mathcal{H})$. The following are equivalent:*

i. ϕ is completely positive;

ii. there exist a representation $(\rho, \mathcal{H}_\rho)$ of A, and a bounded operator $V : \mathcal{H} \to \mathcal{H}_\rho$, such that

$$\phi(a) = V^* \rho(a) V, \quad a \in A.$$

□

1.1.3 Hilbert modules

Let A be a C^*-algebra. At several points we will make use of spaces equipped with an A-valued 'inner product'. We will not require much theory about such spaces so simply record the main definition, an example, and refer to Lance [40] for the general theory.

Let A be a C^*-algebra and E a vector space. Then E is called a *right module over A* if there exists a map $\cdot : E \times A \to E$ satisfying

i. $x \cdot (a + b) = x \cdot a + x \cdot b$ for all $x \in E,\ a, b \in A$;

ii. $(x + y) \cdot a = x \cdot a + y \cdot a$ for all $x, y \in E,\ a \in A$;

iii. $x \cdot (ab) = (x \cdot a) \cdot b$ for all $x \in E,\ a, b \in A$;

iv. $\lambda(x \cdot a) = (\lambda x) \cdot a = x \cdot (\lambda a)$ for all $\lambda \in \mathbb{C},\ x \in E,\ a \in A$.

Similarly we define a *left module* and *bimodule* over A.

Definition 1.24. Let A be a C^*-algebra. An *inner product A-module* is a right A-module E, together with a map

$$\langle \cdot | \cdot \rangle : E \times E \to A,$$

satisfying the conditions:

i. $\langle x | \alpha y + \beta z \rangle = \alpha \langle x | y \rangle + \beta \langle x | z \rangle$ for all $x, y, z \in E,\ \alpha, \beta \in \mathbb{C}$;

ii. $\langle x|ya\rangle = \langle x|y\rangle\, a$ for all $x, y \in E,\ a \in A$;

iii. $\langle x|y\rangle = \langle y|x\rangle^*$ for all $x, y \in E$;

iv. $\langle x|x\rangle \geq 0$ for all $x \in E$, and if $\langle x|x\rangle = 0$ then $x = 0$.

For $x \in E$ we let $\|x\| := \|\, \langle x|x\rangle\, \|^{1/2}$, which defines a norm on E which is compatible with the module structure. An inner product A-module which is complete in this norm is called a *Hilbert C^*-module over A*, or a *Hilbert A-module*. A Hilbert A-module which is also an A-bimodule is called a *Hilbert A-bimodule*.

A Hilbert A-bimodule E is called *countably generated* if there exists a countable set $(x_i)_{i\in\mathbb{N}} \subseteq E$ such that the space of finite sums of the x_i with coefficients in A is dense in E.

I have defined the inner product to be conjugate-linear in the first variable and linear in the second variable. Although there is an unfortunate difference between Hilbert spaces and Hilbert $\mathbb{C}$-modules there would be no difficulty if the definition was changed to correct this; I have chosen to follow the convention used by Lance.

Example 1.25. Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a C^*-algebra, and let $(\rho, \mathcal{H}_\rho)$ be a faithful representation of A. Then $\mathcal{B}(\mathcal{H}, \mathcal{H}_\rho)$ is an A-bimodule under the operations

$$\begin{aligned} a \cdot T &:= \rho(a)T, \\ T \cdot a &:= Ta, \end{aligned}$$

for $a \in A,\ T \in \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho)$. Suppose further that $\Psi : \mathcal{B}(\mathcal{H}) \to A$ is a contractive, completely positive, projection (*i.e.* Ψ is a *conditional expectation* from $\mathcal{B}(\mathcal{H})$ to A — see Brown–Ozawa [11, Theorem 1.5.10]) which is *faithful*, in the sense that $\Psi(x) = 0$ implies $x = 0$ for all positive $x \in \mathcal{B}(\mathcal{H})$. Then the map defined by

$$\langle S|T\rangle := \Psi(S^*T), \quad S, T \in \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho),$$

endows $\mathcal{B}(\mathcal{H}, \mathcal{H}_\rho)$ with an inner product A-module structure; see Lance [40, page 7].

1.1.4 Tensor products

Here we describe tensor products of some spaces introduced in this section.

Let E and F be vector spaces and write $E \odot F$ for the *algebraic tensor product of* E *and* F. The algebraic tensor product is spanned by *elementary tensors* $x \otimes y$, where $x \in X$ and $y \in Y$; I will write a typical element $v \in E \odot F$ by $v = \sum_{i=1}^{n} x_i \otimes y_i$ $(x_i \in X,\ y_i \in Y)$. Elementary tensors satisfy:

$$
\begin{aligned}
(x_1 + x_2) \otimes y &= x_1 \otimes y + x_2 \otimes y \\
x \otimes (y_1 + y_2) &= x \otimes y_1 + x \otimes y_2 \\
\lambda(x \otimes y) &= (\lambda x) \otimes y = x \otimes (\lambda y)
\end{aligned}
$$

for all $x, x_1, x_2 \in E,\ y, y_1, y_2 \in F,\ \lambda \in \mathbb{C}$.

If E and F are algebras then $E \odot F$ has an algebra structure, given by

$$\left(\sum_{i=1}^{n} x_i \otimes y_i\right)\left(\sum_{j=1}^{m} w_j \otimes z_j\right) := \sum_{i,j} x_i w_j \otimes y_i z_j, \quad x_i, w_j \in E,\ y_i, z_j \in F.$$

If E and F both have involutions, denoted by $*$, then $E \odot F$ carries an involution given by

$$\sum_{i=1}^{n} x_i \otimes y_i \mapsto \sum_{i=1}^{n} x_i^* \otimes y_i^*, \quad x_i \in E,\ y_i \in F.$$

There is a canonical bilinear map

$$q : E \times F \to E \odot F;\ q(x, y) = x \otimes y, \quad x \in E,\ y \in F.$$

The algebraic tensor product has the following universal property: for any vector space X and any bilinear map $\phi : E \times F \to X$ there exists a unique linear map $\phi' : E \odot F \to X$ such that $\phi' \circ q = \phi$. These facts and further discussion can be found in Brown–Ozawa [11, Section 3.1].

Each tensor product in this section will be a completion of the algebraic tensor product under a suitably defined norm, so that the space of finite sums of elementary tensors forms a dense subspace.

1.1.4.1 Hilbert spaces

Let $\mathcal{H}$ and $\mathcal{L}$ be Hilbert spaces. On the algebraic tensor product $\mathcal{H} \odot \mathcal{L}$ we define an inner product by

$$\left\langle \sum_{i=1}^{n} h_i \otimes k_i, \sum_{j=1}^{m} h_j' \otimes k_j' \right\rangle := \sum_{i,j} \langle h_i, h_j' \rangle \langle k_i, k_j' \rangle, \quad h_i, h_j' \in \mathcal{H},\ k_i, k_j' \in \mathcal{L}.$$

The completion of $\mathcal{H} \odot \mathcal{L}$ in the norm induced by this inner product (1.1) is a Hilbert space, called the *Hilbert space tensor product*, denoted $\mathcal{H} \otimes \mathcal{L}$.

1.1.4.2 Algebras of operators

The definitions given here can all be found in Brown–Ozawa [11, Chapter 3]. Let $\mathcal{H}$ and $\mathcal{L}$ be Hilbert spaces. Identifying $\mathcal{B}(\mathcal{H})$ with $\mathcal{B}(\mathcal{H}) \otimes \mathbb{C}I_{\mathcal{L}}$ and $\mathcal{B}(\mathcal{L})$ with $\mathbb{C}I_{\mathcal{H}} \otimes \mathcal{B}(\mathcal{L})$, and using the fact that these two subalgebras of $\mathcal{B}(\mathcal{H} \otimes \mathcal{L})$ commute [11, Proposition 3.1.17], we obtain a $*$-homomorphism from $\mathcal{B}(\mathcal{H}) \odot \mathcal{B}(\mathcal{L})$ into $\mathcal{B}(\mathcal{H} \otimes \mathcal{L})$ sending an elementary tensor $S \otimes T \in \mathcal{B}(\mathcal{H}) \odot \mathcal{B}(\mathcal{L})$ to the operator[1]

$$S \otimes T : \sum_{i=1}^{n} h_i \otimes k_i \mapsto \sum_{i=1}^{n} Sh_i \otimes Tk_i, \quad \sum_{i=1}^{n} h_i \otimes k_i \in \mathcal{H} \otimes \mathcal{L}.$$

Definition 1.26. Let A and B be C^*-algebras. Choose faithful representations $(\rho, \mathcal{H}_\rho)$ and $(\psi, \mathcal{H}_\psi)$ of A and B respectively. The *spatial*, or *minimal*, C^*-norm on $A \odot B$ is defined by

$$\left\| \sum_{i=1}^{n} a_i \otimes b_i \right\|_{\min} := \left\| \sum_{i=1}^{n} \rho(a_i) \otimes \psi(b_i) \right\|, \quad \sum_{i=1}^{n} a_i \otimes b_i \in A \odot B,$$

[1] Note the abuse of the symbol $\otimes$ here.

where the norm on the right is the norm of $\mathcal{B}(\mathcal{H}_\rho \otimes \mathcal{H}_\psi)$. The completion of $A \odot B$ under this norm is a C^*-algebra, called the *minimal (or spatial) tensor product of A and B*, and denoted $A \otimes_{\min} B$.

The spatial norm is independent of the choice of faithful representations [11, Proposition 3.3.11], and is in fact the smallest possible C^*-norm on the algebraic tensor product of C^*-algebras [11, Section 3.4].

For a C^*-algebra A faithfully represented on a Hilbert space $\mathcal{H}$ one can show that the space $M_n(A)$ used in Subsection 1.1.2 is in fact $M_n \odot A$, and the C^* norm it inherits from $\mathcal{B}(\mathcal{H}^n)$ is unique; see Brown–Ozawa [11, Proposition 3.3.2]. It will occasionally be useful that a map ϕ on A is completely bounded if and only if the norms of $\mathrm{id}_{M_n} \otimes \phi$ are uniformly bounded.

We will also need to use the tensor product of von Neumann algebras.

Definition 1.27. Let M and N be von Neumann algebras on Hilbert spaces $\mathcal{H}_M$ and $\mathcal{H}_N$ respectively. The *weak* spatial tensor product of M and N*, denoted $M \overline{\otimes} N$, is the von Neumann algebra generated by $M \otimes \mathbb{C}I_{\mathcal{H}_N}$ and $I_{\mathcal{H}_M} \otimes N$ in $\mathcal{B}(\mathcal{H}_M \otimes \mathcal{H}_N)$; that is, the weak closure of $M \otimes_{\min} N$ in $\mathcal{B}(\mathcal{H}_M \otimes \mathcal{H}_N)$.

1.1.4.3 Operator spaces

We will make use of several operator space tensor products. Let us begin by introducing the injective operator space tensor product, which is defined in the same way as the minimal tensor product of C^*-algebras; see Brown–Ozawa [11, Remark 3.3.6] and Effros–Ruan [22, Proposition 8.1.6].

Definition 1.28. Let V and W be operator spaces. Use Theorem 1.20 to find Hilbert spaces $\mathcal{H}_\Phi$ and $\mathcal{H}_\Psi$ and complete isometries $\Phi : V \to \mathcal{B}(\mathcal{H}_\Phi)$ and $\Psi : W \to \mathcal{B}(\mathcal{H}_\Psi)$. Define a norm $\|\cdot\|_{\min}$ on $V \odot W$ by

$$\left\| \sum_{i=1}^n x_i \otimes y_i \right\|_{\min} := \left\| \sum_{i=1}^n \Phi(x_i) \otimes \Psi(y_i) \right\|, \quad \sum_{i=1}^n x_i \otimes y_i \in V \odot W,$$

where the norm on the right side is the norm of $\mathcal{B}(\mathcal{H}_\Phi \otimes \mathcal{H}_\Psi)$. The completion of $V \odot W$ under $\|\cdot\|_{\min}$ is an operator space called the *minimal, or injective, operator space tensor product of V and W*, and denoted $V \otimes_{\min} W$.

Note that the operator space injective tensor product of V and W can also be described as the completion of $V \odot W$ under the norm defined for $u \in M_n(V \odot W)$ by

$$\|u\| := \sup\{\|(f \otimes g)^{(n)}(u)\| : f \in M_p(V^*),\ g \in M_q(W^*),\ \|f\|, \|g\| \leq 1\};$$

indeed this is the definition used by Effros–Ruan [22, page 138]. It is clear from Definition 1.28 that the injective operator space tensor product of two C^*-algebras coincides with the minimal tensor product of C^*-algebras, so there is no danger of confusion arising from the use of the subscript min for both.

The weak* spatial tensor product of operator spaces is defined similarly to that of von Neumann algebras; see the discussion given by Effros–Ruan [22, Theorem 7.2.3] for further information.

Definition 1.29. Let V and W be operator spaces which are the dual spaces of some complete operator spaces. Let ϕ and ψ be *dual realisations* of V and W respectively; that is, there exist Hilbert spaces $\mathcal{H}_\phi$ and $\mathcal{H}_\psi$ such that $\phi : V \to \mathcal{B}(\mathcal{H}_\phi)$ and $\psi : W \to \mathcal{B}(\mathcal{H}_\psi)$ are completely isometric, injective, weak* homeomorphisms [22, Proposition 3.2.4]. The *normal spatial tensor product of V and W* is defined to be the closure of $V \odot W$ in the weak* topology on $\mathcal{B}(\mathcal{H}_\phi \otimes \mathcal{H}_\psi)$, and denoted by $V \overline{\otimes} W$.

It is clear from the definitions that the above reduces to Definition 1.27 when the operator spaces involved are von Neumann algebras, so that there is no possibility of confusion arising from the symbol $\overline{\otimes}$.

We refer to Effros–Ruan [22, Chapters 7 and 8] for an exposition of the theory of these tensor products, including the elegant interplay with duality and completely bounded maps.

There is another operator space tensor product, called the *Haagerup tensor product*, which will be used in Section 4.2. Since the Haagerup tensor product will not appear anywhere else in this thesis we postpone the definition until Section 4.2 where it is required.

1.2 Measure and integration

We will make extensive use of measure theory and integrals. In this section I give basic definitions and facts which will be used throughout, as well as serving as background for Section 1.5. We must take care to ensure the definitions and statements in this section agree with the requirements of Section 1.5. The material is based on Cohn [13], Rudin [50, Chapter 2], and Williams [60, Appendix B].

Some of our statements will use the extended positive real numbers $[0, \infty]$; see Rudin [50, page 18] for a discussion of this space and its properties.

Definition 1.30. Let X be a set. A collection $\mathscr{S}$ of subsets of X is called a *σ-algebra* if it satisfies the following conditions:

i. $X \in \mathscr{S}$;

ii. $\mathscr{S}$ is closed under complements;

iii. $\mathscr{S}$ is closed under countable unions;

iv. $\mathscr{S}$ is closed under countable intersections.

A *measure* on $(X, \mathscr{S})$ is a function $\mu : \mathscr{S} \to [0, \infty]$ which is countably additive and satisfies $\mu(\emptyset) = 0$. A *measure space* is a triple $(X, \mathscr{S}, \mu)$ where X is a set, $\mathscr{S}$ a σ-algebra on X, and μ a measure on $(X, \mathscr{S})$.

A measure μ (or a measure space $(X, \mathscr{S}, \mu)$) is called *finite* if $\mu(X)$ is finite and *σ-finite* if X is the union of a countable family of sets, each of which has finite measure.

I will often write "let (X, μ) be a measure space", or "let μ be a measure on X", in place of "let $(X, \mathscr{S}, \mu)$ be a measure space". A set of measure 0 will be called *null*. A set N will be called *locally null* if $A \cap N$ is null for every set A of finite measure; locally null sets will be used to define $L^\infty(X)$ below, but we will mostly be working with σ-finite measure spaces, in which case locally null and null are equivalent by Cohn [13, page 92].

Now to discuss measurable and integrable functions.

Definition 1.31. Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \nu)$ be measure spaces and $f : X \to Y$ a function. Then f is called *measurable with respect to $\mathscr{S}$ and $\mathscr{T}$* if for every $B \in \mathscr{T}$ we have $f^{-1}(B) \in \mathscr{S}$. For the definition of the integral of a scalar valued function we refer to Cohn [13, Chapter 2].

When the σ-algebras are clear from context I will simply call a function $f : X \to Y$ measurable.

Associated to a measure space are a number of Banach spaces; here we give the definition, from Cohn [13, Section 3.3], and refer to Cohn [13, Sections 3.3 and 3.4] for the properties of such spaces.

Definition 1.32. Let (X, μ) be a measure space and $1 \leq p < \infty$. Let $\mathfrak{L}^p(X)$ denote the collection of all measurable functions $f : X \to \mathbb{C}$ such that

$$\|f\|_p := \left(\int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}$$

is well-defined and finite. This space is a vector space on which $\|\cdot\|_p$ defines a seminorm. Let $\mathcal{N}^p(X)$ denote the space of functions f for which $\|f\|_p = 0$. The quotient space

$$L^p(X) := \mathfrak{L}^p(X)/\mathcal{N}^p(X)$$

is a Banach space under the quotient norm, which is again denoted by $\|\cdot\|_p$.

We define $\mathfrak{L}^\infty(X)$ to be the space of all measurable functions $f : X \to \mathbb{C}$ such that

$$\|f\|_\infty := \inf \left\{ M \geq 0 : \{x \in X : |f(x)| > M\} \text{ is locally null} \right\} \tag{1.3}$$

is finite. The quotient space of $\mathfrak{L}^\infty(X)$ formed by identifying functions which agree locally almost everywhere is a Banach space, denoted $L^\infty(X)$, under the quotient norm, which is again written $\|\cdot\|_\infty$.

In this thesis we will only make use of $L^1(X)$, $L^2(X)$, and $L^\infty(X)$. I will follow the custom of treating elements of $L^p(X)$ as functions defined almost everywhere rather than equivalence classes.

To each $f \in L^\infty(X)$ we associate the operator

$$M_f : L^2(X) \to L^2(X);\ M_f\xi(x) := f(x)\xi(x), \quad x \in X,\ \xi \in L^2(X).$$

It is easy to see that $M_f \in \mathcal{B}(L^2(X))$. The algebra of operators M_f associated to functions $f \in L^\infty(X)$ will be denoted $\mathcal{D}_X$.

Let us state the central result of product measures, given by Cohn [13, Theorem 5.1.4]. Product measures will be used throughout this thesis; we refer to Cohn [13, Chapter 5] for further properties and discussion.

Theorem 1.33. *Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \nu)$ be σ-finite measure spaces. The σ-algebra on $X \times Y$ generated by sets of the form $A \times B$, with $A \in \mathscr{S}$ and $B \in \mathscr{T}$, is called the* product σ-algebra of $\mathscr{S}$ and $\mathscr{T}$*, denoted $\mathscr{S} \times \mathscr{T}$. There exists a unique measure $\mu \times \nu$ on $\mathscr{S} \times \mathscr{T}$ satisfying*

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B), \quad A \in \mathscr{S},\ B \in \mathscr{T}.$$

The measure $\mu \times \nu$ is called the product *of μ and ν.* □

Some properties of product measures and integrals will be given in the vector-valued case in Theorem 1.75.

We will mainly be considering measure and integration on spaces with a Hausdorff topology. In this case there is a natural σ-algebra, the existence of which is guaranteed by Cohn [13, Corollary 1.1.3].

Definition 1.34. Let X be a Hausdorff topological space. The σ-algebra generated by the open (equivalently, closed) subsets of X is called the *Borel σ-algebra on X*, and denoted $\mathscr{B}(X)$. The elements of $\mathscr{B}(X)$ are called *Borel sets*.

If a space X is locally compact and Hausdorff measures on X are linked to linear functionals on $C_c(X)$ *via* the following result, called the *Riesz Representation Theorem*; a proof is given by Rudin [50, Theorem 2.14]. This is precisely the statement required by Williams [60, Section B.1], and is necessary for Section 1.5.

Theorem 1.35. *Let X be a locally compact Hausdorff space, and let Λ be a positive linear functional on $C_c(X)$; that is, if $f \in C_c(X)$ satisfies $f(X) \subset [0, \infty)$ then $\Lambda f \in [0, \infty)$. Then there exists a σ-algebra $\mathscr{M}$ in X which contains the Borel sets of X, and a unique measure μ on $(X, \mathscr{M})$, satisfying the conditions:*

i. $\Lambda f = \int_X f(x)\, d\mu(x)$ *for every* $f \in C_c(X)$;

ii. $\mu(K) < \infty$ *for every compact set* $K \subseteq X$;

iii. *for every* $E \in \mathscr{M}$ *we have*

$$\mu(E) = \inf\{\mu(V) : E \subseteq V,\ V \textit{ is open}\};$$

iv. *the relation*

$$\mu(E) = \sup\{\mu(K) : K \subseteq E,\ K \textit{ is compact}\}$$

holds for every open set E, *and for every* $E \in \mathscr{M}$ *with* $\mu(E) < \infty$;

v. *if* $E \in \mathscr{M}$, $A \subseteq E$, *and* $\mu(E) = 0$, *then* $A \in \mathscr{M}$. □

Definition 1.36. Let X be a locally compact, Hausdorff, topological space. A *Radon measure on X* is a measure on $(X, \mathscr{M})$ arising from a positive linear functional on $C_c(X)$ by Theorem 1.35.

We refer to Williams [60, Appendix B] for further discussion of the consequences of this definition, and how it compares to classical treatments of (vector-valued) integration on locally compact spaces.

Definition 1.37. A measure space $(X, \mathscr{S}, \mu)$ is called *standard* if there exists a separable, metrizable, topological space Y and a bijection $f : X \to Y$ such that f is measurable with respect to $\mathscr{S}$ and $\mathscr{B}(Y)$ and f^{-1} is measurable with respect to $\mathscr{B}(Y)$ and $\mathscr{S}$.

When working with standard measure spaces we will often make the additional assumption that the underlying topology is locally compact.

1.3 Locally compact groups

Much of this thesis will be concerned with functions defined on groups. In this section we build up the theory of locally compact topological groups, and their associated Banach spaces. The identity element of a group will usually be denoted by e. We follow Williams [60, Chapter 1] in requiring that our topological groups be T_1 spaces.

Definition 1.38. A *topological group* is a group G together with a topology τ on G such that:

i. points are closed in (G, τ);

ii. the group operations are continuous.

An equivalent condition to (ii) above is that the map

$$G \times G \to G;\ (s, t) \mapsto st^{-1}$$

is continuous.

The following examples will be used throughout this thesis.

Examples 1.39. (i) Any group with the discrete topology is a topological group; such groups will be called *discrete groups*.
(ii) Let $\mathcal{H}$ be a Hilbert space. Then the group of unitary operators on $\mathcal{H}$, $\mathcal{U}(\mathcal{H})$, defined by

$$\mathcal{U}(\mathcal{H}) := \{U \in \mathcal{B}(\mathcal{H}) : U^*U = UU^* = I_{\mathcal{H}}\},$$

endowed with the strong operator topology, is a topological group.

(iii) Let A be a C^*-algebra. Then $\mathrm{Aut}(A)$, the collection of ($*$-)automorphisms of A, is a group with the group operation being composition. It becomes a topological group when endowed with the *point-norm topology*: a net $(\alpha_\gamma)_{\gamma\in\Gamma} \subset \mathrm{Aut}(A)$ converges to $\alpha \in \mathrm{Aut}(A)$ if and only if $\alpha_\gamma(a) \to \alpha(a)$ for all $a \in A$.

It can be shown that a topological group automatically has a Hausdorff topology which is regular; that is, given a closed set X and $t \notin X$ there exist disjoint open sets separating t and X [60, Lemma 1.13].

Definition 1.40. A *locally compact group* is a topological group for which the underlying topology is locally compact.

An important fact about locally compact groups is the automatic existence of a useful measure. The results summarised by the following theorem can be found, for example, in Folland [24, Section 2.2].

Theorem 1.41. *Every locally compact group has a left-invariant Radon measure which is unique up to a strictly positive scalar. Such a measure is called a* (left) Haar measure*.*

□

Let G be a locally compact group with a left Haar measure m. Then, for any $t \in G$, the measure m_t on G defined by

$$m_t(E) := m(Et), \quad E \subseteq G,$$

is another left Haar measure on G. By Theorem 1.41 there is a positive real number, denoted $\Delta(t)$, such that $m_t = \Delta(t)m$; moreover, $\Delta(t)$ is independent of the original choice of m. Let $\mathbb{R}_{>0}$ denote the multiplicative group of positive real numbers. The function $\Delta : G \to \mathbb{R}_{>0}$ is called the *modular function* of G; in fact it is a continuous homomorphism from G to $\mathbb{R}_{>0}$ [24, Proposition 2.24]. Throughout this work every locally compact group will have a fixed left Haar measure. Integration, over the variable s, with respect to Haar measure will be written ds.

We will often assume that our groups are second-countable, so Proposition 1.3 implies that there is an increasing sequence of compact subsets $(K_n)_{n\in\mathbb{N}}$ of G such that $G = \cup_{n\in\mathbb{N}} K_n$. Moreover, it is a standard fact that second-countable, locally compact, Hausdorff spaces are metrizable [13, Proposition 7.1.15], so that second-countable, locally compact groups are standard measure spaces when equipped with a Haar measure. The following remark will be used several times.

Remark 1.42. Suppose that G is a second-countable, locally compact group. Then $L^2(G)$ is separable. Indeed, if G is second-countable then Haar measure m on G is σ-finite by Proposition 1.3, and the second-countability also implies that the σ-algebra on which m is defined is countably generated. The claim now follows from Cohn [13, Proposition 3.4.5]. □

1.3.1 Representations and positive-definite functions

There is a neat relationship between unitary representations of a locally compact group, representations of its associated group C^*-algebra, and certain 'positive' functions on the group.

Definition 1.43. Let G be a locally compact group. A *unitary representation* τ of G on a Hilbert space $\mathcal{H}_\tau$ is a homomorphism $\tau : G \to \mathcal{U}(\mathcal{H}_\tau)$ which is continuous in the strong operator topology; that is, for any $s \in G$ and any $\xi \in \mathcal{H}_\tau$ the map

$$s \mapsto \tau(s)\xi$$

is a continuous function from G to $\mathcal{H}_\tau$. We say τ is *irreducible* if $\mathcal{H}_\tau \neq \{0\}$ and the only closed subspaces of $\mathcal{H}_\tau$ invariant under τ_G are $\{0\}$ and $\mathcal{H}_\tau$.

Note that for any Hilbert space $\mathcal{H}$ the strong and weak operator topologies coincide on $\mathcal{U}(\mathcal{H})$, so it is equivalent to define a unitary representation of G on $\mathcal{H}$ to be a homomorphism from G to $\mathcal{U}(\mathcal{H})$ continuous in the weak operator topology [24, page 68]. Since no other representations of a group will be used I will sometimes refer simply to a representation of the group, rather than a unitary representation. If τ is a representation of G I will often write τ_s in place of $\tau(s)$.

Example 1.44. Let G be a locally compact group. Then

$$\lambda^G : G \to \mathcal{B}(L^2(G));\ (\lambda^G_s \xi)(t) := \xi(s^{-1}t), \quad s, t \in G,\ \xi \in L^2(G),$$

is a unitary representation of G on $L^2(G)$ called the *left regular representation.*

The following result summarises the relevant discussion of Dixmier [18, Section 13.3]. Note the vector-valued integral appearing in equation (1.4) below. A suitable theory of such integrals will be given in Section 1.5; alternatively, one can replace $L^1(G)$ with $C_c(G)$ in this section and use the simpler theory given by Williams [60, Section 1.5] which suffices here.

Proposition 1.45. *Let G be a locally compact group. There is a bijective correspondence between unitary representations of G and non-degenerate representations of $L^1(G)$. If τ is a representation of G on the Hilbert space $\mathcal{H}_\tau$ then the associated representation of $L^1(G)$ is given by*

$$\tau(f) := \int_G f(s)\tau_s \, ds, \quad f \in L^1(G). \tag{1.4}$$

Moreover, τ is irreducible if and only if the associated representation of $L^1(G)$ is irreducible. ☐

The collection of all unitary representations of a locally compact group G, which by the above Proposition is the same as the collection of all non-degenerate representations of $L^1(G)$, will be denoted by $\Sigma(G)$ or simply Σ if there can be no confusion. Let $S \subseteq \Sigma$; the set

$$N_S := \{f \in L^1(G) : \tau(f) = 0 \text{ for all } \tau \in S\},$$

is a closed, two-sided ideal of $L^1(G)$. Writing $\dot{f}$ for the equivalence class of $f \in L^1(G)$ in $L^1(G)/N_S$ we have

$$\|\dot{f}\|_S := \sup_{\tau \in S} \|\tau(f)\|$$

is independent of the representative of $\dot{f}$ and defines a C^*-norm on $L^1(G)/N_S$.

I will make an effort to reduce clutter in the notation when writing the subscript S; for example, if $S = \{\lambda^G\}$ the subscript λ will be used.

Definition 1.46. Let G be a locally compact group and $S \subseteq \Sigma$. The completion of $L^1(G)/N_S$ in $\|\cdot\|_S$ is a C^*-algebra denoted $C^*_S(G)$.

In particular,

$$\|f\|_\Sigma := \sup_{\tau \in \Sigma} \|\tau(f)\|.$$

The completion of $L^1(G)$ in $\|\cdot\|_\Sigma$ is a C^*-algebra called the *group C^*-algebra of G* and denoted $C^*(G)$.

We also have

$$\|f\|_\lambda := \|\lambda^G(f)\|.$$

The completion of $L^1(G)$ in $\|\cdot\|_\lambda$ is a C^*-algebra called the *reduced group C^*-algebra of G* and denoted $C^*_\lambda(G)$.

The subscript on the norms defined above will be omitted if it is clear from context.

By Dixmier [18, Proposition 2.7.4], $L^1(G)$ has the same non-degenerate representations as its enveloping C^*-algebra $C^*(G)$. It follows from Proposition 1.45 that the unitary representations of G and representations of $C^*(G)$ are in bijective correspondence; moreover this correspondence preserves irreducibility. If τ is a unitary representation of G, τ_G and $\tau(L^1(G))$ have the same commutant in $\mathcal{B}(\mathcal{H}_\tau)$ and therefore generate the same von Neumann algebra [18, 13.3.5]. The most commonly used such is that associated to the left regular representation.

Definition 1.47. Let G be a locally compact group. The *group von Neumann algebra of G*, denoted $\mathrm{vN}(G)$, is the von Neumann algebra generated in $\mathcal{B}(L^2(G))$ by λ^G_G, or equivalently $\lambda^G(L^1(G))$. Thus $\mathrm{vN}(G) = C^*_\lambda(G)''$.

We seek a notion of 'positivity' for functions on a group which lifts to give a positive linear functional on $L^1(G)$.

Definition 1.48. Let G be a locally compact group. A function $u : G \to \mathbb{C}$ is called *positive-(semi)definite* if, for any $n \in \mathbb{N}$ and any $s_1, \ldots, s_n \in G$, the matrix

$$\big(u(s_j s_i^{-1})\big)_{i,j=1}^n \tag{1.5}$$

is positive (or 0).

An equivalent condition to (1.5) is that for any $c_1, \ldots, c_n \in \mathbb{C}$ we have

$$\sum_{i,j=1}^{n} \overline{c_i} c_j u(s_j s_i^{-1}) \geq 0.$$

If G is a locally compact group, τ a representation of G, and $\xi \in \mathcal{H}_\tau$, then it is easily seen that

$$u_{\tau,\xi} : G \to \mathbb{C};\ u_{\tau,\xi}(s) := \langle \tau_s \xi, \xi \rangle\,, \quad s \in G,$$

is a positive-definite function on G, called the positive-definite function *associated to* τ *and* ξ; allowing ξ to vary we obtain the positive-definite functions associated to τ. If $S \subseteq \Sigma$ then we let $\mathcal{P}_S(G)$ be the set of all positive-definite functions associated to the elements of S.

The following Proposition of Eymard [23, Proposition 1.15] clarifies the relationship between the C^*-algebras $C^*_S(G)$ and $C^*(G)$.

Proposition 1.49. *Let G be a locally compact group and $S \subseteq \Sigma$. Let also N'_S denote the intersection of the kernels of elements of S acting as representations of $C^*(G)$. The quotient map $f \mapsto \dot{f}$ from $L^1(G)$ to $L^1(G)/N_S$ extends to a norm-reducing homomorphism $q_S : C^*(G) \to C^*_S(G)$ with kernel N'_S.* □

1.3.2 Fourier and Fourier–Stieltjes algebras

The Fourier and Fourier–Stieltjes algebras of a locally compact group will be useful. The following Proposition is due to Eymard [23, Proposition 2.1]; for the meaning of weak containment in condition (ii) see Dixmier [18, 3.4.5 and 18.1.3].

Proposition 1.50. *Let G be a locally compact group, $u : G \to \mathbb{C}$, and $S \subseteq \Sigma$. The following are equivalent:*

i. u is a finite linear combination of elements of $\mathcal{P}_S(G)$;

ii. there is $\tau \in \Sigma$, weakly contained in S, and $\xi, \eta \in \mathcal{H}_\tau$ such that $u(t) = \langle \tau_t \xi, \eta \rangle$ for all $t \in G$;

iii. u is bounded and continuous on G and

$$\|u\|_S := \sup_{f \in L^1(G), \|f\|_S \leq 1} \left| \int_G f(s)u(s)\,ds \right| < \infty. \qquad \square$$

Definition 1.51. Let G be a locally compact group. We denote by $B_S(G)$ the space of all functions satisfying the equivalent conditions of Proposition 1.50. This space is a Banach algebra under $\|\cdot\|_S$ and pointwise multiplication. Eymard [23, Lemme 2.14] shows that

$$\|u\|_S = \inf\{\|\xi\|\|\eta\| : u(t) = \langle \tau_t \xi, \eta \rangle \ \ (t \in G)\}.$$

The *Fourier–Stieltjes algebra* of G is $B(G) := B_\Sigma(G)$, which by (ii) above is the space of coefficients of unitary representations of G. We will also make use of the *reduced Fourier–Stieltjes algebra* $B_\lambda(G)$.

The Fourier–Stieltjes algebra of a locally compact group G can be identified with the dual of $C^*(G)$, the duality given by

$$\begin{aligned} u(f) &= \int_G f(s)u(s)\,ds, \quad f \in L^1(G),\ u \in B(G),\ \text{or} \\ u(x) &= \langle \tau(x)\xi, \eta \rangle, \quad x \in C^*(G),\ u(\cdot) = \langle \tau(\cdot)\xi, \eta \rangle \in B(G). \end{aligned}$$

In a similar way the reduced Fourier–Stieltjes algebra of G can be identified with the dual of the reduced group C^*-algebra.

The following summarises some results of Eymard [23, Chapitre 3] which we use to define the Fourier algebra.

Proposition 1.52. *Let G be a locally compact group. The closure of*

$$B(G) \cap C_c(G) = \operatorname{span}\left(\mathcal{P}(G) \cap C_c(G)\right)$$

in $B(G)$ is equal to the space

$$\{u : G \to \mathbb{C} : u(\cdot) = \left\langle \lambda^G_\cdot \xi, \eta \right\rangle,\ \xi, \eta \in L^2(G)\}$$

of coefficients of the left regular representation. The resulting space is a Banach algebra under pointwise multiplication and the norm inherited from $B(G)$; moreover, it is an ideal in $B(G)$. □

Definition 1.53. The Banach algebra characterised in Proposition 1.52 is called the *Fourier algebra* of G, denoted $A(G)$.

When G is an abelian group the Fourier algebra can be identified with the Fourier transform of $L^1(\hat{G})$, where $\hat{G}$ denotes the dual group of G. Eymard [23, Théorème 3.10] shows that the Fourier algebra of a locally compact group G is the (unique) predual of the group von Neumann algebra, *i.e.* $A(G) = \mathrm{vN}(G)_*$.

1.4 Schur and Herz–Schur multipliers

This section lays out the known theory of 'classical' Schur and Herz–Schur multipliers, which we aim to generalise. For this reason we set out the basic theory in some detail, in a way that is convenient for generalisation later. A concise reference for the material in this section is the lecture notes of Todorov [56].

1.4.1 Schur multipliers

Here we define Schur multipliers as those functions for which pointwise multiplication of the integral kernel defines a bounded map on the space of Hilbert–Schmidt operators. This straightforward idea, which clearly generalises Schur multiplication of matrices, is central to the definition of a Schur multiplier presented here and in Chapter 2; however we wish to study Schur multipliers defined on measure spaces, which involves many subtleties.

Let us begin by defining two subsets of the compact operators between Hilbert spaces. We refer to Weidmann [59] for the definitions and facts stated below.

Definition 1.54. Let $\mathcal{H}$ and $\mathcal{L}$ be Hilbert spaces. For an operator $S \in \mathcal{B}(\mathcal{H})$ we define the *trace of* S by

$$\operatorname{tr}(S) := \sum_{i \in I} \langle S e_i, e_i \rangle ,$$

where $(e_i)_{i \in I}$ is an orthonormal basis for $\mathcal{H}$.

Now suppose $T \in \mathcal{B}(\mathcal{H}, \mathcal{L})$, so that $T^*T \in \mathcal{B}(\mathcal{H})$, and define

$$\|T\|_1 := \operatorname{tr}\left((T^*T)^{\frac{1}{2}}\right);$$

the operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ is said to be *of trace class* if $\|T\|_1$ is finite for some choice of orthonormal basis $(e_i)_{i \in I}$; in this case $\|T\|_1$ is independent of the choice of orthonormal basis. We denote the space of all trace class operators from $\mathcal{H}$ to $\mathcal{L}$ by $\mathcal{S}_1(\mathcal{H}, \mathcal{L})$. Define also

$$\|T\|_2 := \operatorname{tr}(T^*T)^{\frac{1}{2}};$$

the operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ is said to be a *Hilbert–Schmidt operator* if $\|T\|_2$ is finite. We denote the space of all Hilbert–Schmidt operators from $\mathcal{H}$ to $\mathcal{L}$ by $\mathcal{S}_2(\mathcal{H}, \mathcal{L})$.

The trace class operators $\mathcal{S}_1(\mathcal{H}, \mathcal{L})$ can be identified with the predual of $\mathcal{B}(\mathcal{L}, \mathcal{H})$, with the duality given by

$$\langle T, S \rangle := \operatorname{tr}(ST) = \operatorname{tr}(TS), \quad S \in \mathcal{S}_1(\mathcal{H}, \mathcal{L}),\ T \in \mathcal{B}(\mathcal{L}, \mathcal{H}).$$

Let (X, μ) and (Y, ν) be standard measure spaces. We write $\mathcal{S}_1(X, Y)$ in place of $\mathcal{S}_1(L^2(X), L^2(Y))$ to simplify the notation, similarly for $\mathcal{S}_2(X, Y)$. We will use the following result, given by Weidmann [59, Theorem 6.11].

Theorem 1.55. *Let (X, μ) and (Y, ν) be σ-finite measure spaces. An operator $T \in \mathcal{B}(L^2(X), L^2(Y))$ is a Hilbert–Schmidt operator if and only if there exists $k \in L^2(Y \times X)$, called the integral kernel of T, such that*

$$(T\xi)(y) = \int_X k(y, x)\xi(x)\, d\mu(x), \quad \xi \in L^2(X),$$

almost everywhere. □

We now describe some identifications which will be used to define Schur multipliers. The material presented here is given by Katavalos–Paulsen [36, Section 3]. It is well known that $\mathcal{S}_1(X, Y)$ can be identified with the Banach space projective tensor product $L^2(X) \otimes_\gamma L^2(Y)$ (that is, the completion of $L^2(X) \odot L^2(Y)$ under the greatest crossnorm, see Takesaki [55, Section IV.2]); the identification is given by identifying an elementary tensor $f \otimes g \in L^2(X) \odot L^2(Y)$ with the rank one operator

$$h \mapsto \langle h, \overline{g} \rangle f = \left(\int_Y h(y) g(y) \, d\nu(y) \right) f, \quad h \in L^2(Y).$$

Let us now describe a *pseudotopology* on $X \times Y$ (a pseudotopology on a set Z is a collection of subsets of Z closed under finite intersections and countable unions) which turns out to be the correct way of identifying elements of $L^2(X) \otimes_\gamma L^2(Y)$ with functions on $X \times Y$. A set $E \subseteq X \times Y$ will be called *marginally null* if there exist null sets $M \subseteq X$ and $N \subseteq Y$ such that $E \subseteq (M \times Y) \cup (X \times N)$. Two measurable sets $E, F \subseteq X \times Y$ are called *marginally equivalent* if their symmetric difference is marginally null. A measurable set $E \subseteq X \times Y$ is called ω-*open* if it is marginally equivalent to a set of the form $\cup_{i \in \mathbb{N}} C_i \times D_i$, where $C_i \subseteq X$ and $D_i \subseteq Y$ are measurable sets for all $i \in \mathbb{N}$. The collection of all ω-open sets is a pseudotopology on $X \times Y$ called the ω-*topology.*

Now we identify $L^2(X) \otimes_\gamma L^2(Y)$, and therefore $\mathcal{S}_1(X, Y)$, with a space of functions on $X \times Y$ modulo marginally null sets. Note that some authors consider locally marginally null sets, but we have assumed the measure spaces X and Y are standard, so locally null and null are equivalent by Cohn [13, page 92]. Given $u \in L^2(X) \otimes_\gamma L^2(Y)$ find $f_i \in L^2(X)$ and $g_i \in L^2(Y)$ such that $u = \sum_{i=1}^{\infty} f_i \otimes g_i$, and define

$$\psi_u : X \times Y \to \mathbb{C}; \; \psi_u(x, y) := \sum_{i=1}^{\infty} f_i(x) g_i(y), \quad (x, y) \in X \times Y.$$

The series on the right converges marginally almost everywhere, the identification of functions with elements of the tensor product is well made: $u = 0$ if and only if $\psi_u = 0$ marginally almost everywhere, and ψ_u is independent of the representation of u. We let $T(X, Y)$ denote the space of all functions ψ_u associated to elements $u \in L^2(X) \otimes_\gamma L^2(Y)$, modulo almost everywhere equality. We refer to Arveson [2, Proposition 2.2.7], Katavalos–Paulsen [36, Section 3], and Todorov [56, Section 4] for further details on these identifications.

Definition 1.56. Let $\varphi \in L^\infty(X \times Y)$. The function φ is called a *Schur multiplier* if $\varphi h \in T(X,Y)$ for all $h \in T(X,Y)$; that is, for every $h \in T(X,Y)$ there exists a function $h' \in T(X,Y)$ which is marginally equivalent to φh. The space of Schur multipliers defined on $X \times Y$ is denoted by $\mathfrak{S}(X,Y)$.

For a function $\varphi \in \mathfrak{S}(X,Y)$ define

$$m_\varphi : T(X,Y) \to T(X,Y); \; m_\varphi(h) := \varphi h, \quad h \in T(X,Y).$$

It follows from the Closed Graph Theorem that m_φ is a bounded linear operator on $T(X,Y)$, where the norm on $T(X,Y)$ is the one arising from the identification with $L^2(X) \otimes_\gamma L^2(Y)$. If $h \in T(X,Y)$ then we denote the associated trace class operator by T_h; we have $m_\varphi T_h = T_{\varphi h}$. The norm on $\mathfrak{S}(X,Y)$ is given by $\|\varphi\|_{\mathfrak{S}} := \|m_\varphi\|$. Since $T(X,Y)$ is identified with the predual of $\mathcal{B}(L^2(X), L^2(Y))$ we get a map $S_\varphi := m_\varphi^*$ on $\mathcal{B}(L^2(X), L^2(Y))$. If $h \in T(X,Y)$ let T_h denote the associated trace class operator; also for $k \in L^2(Y \times X)$ let $T_k \in \mathcal{S}_2(X,Y)$ denote the associated Hilbert–Schmidt operator. We have

$$\langle T_k, T_h \rangle = \int_{X \times Y} k(y,x)h(x,y)\, d(\mu \times \nu)(x,y).$$

By definition S_φ is a bounded, weak*-continuous, map; moreover

$$\begin{aligned} \langle S_\varphi(T_k), T_h \rangle &= \langle T_k, m_\varphi(T_h) \rangle \\ &= \langle T_k, T_{\varphi h} \rangle \\ &= \int_{X \times Y} k(y,x)\varphi(x,y)h(x,y)\, d(\mu \times \nu)(x,y) \\ &= \langle T_{\varphi k}, T_h \rangle . \end{aligned} \tag{1.6}$$

Here we have written φk for the function $(y,x) \mapsto \varphi(x,y)k(y,x)$, which is in $L^2(Y \times X)$ because $\varphi \in L^\infty(X \times Y)$. It follows that $S_\varphi(T_k) = T_{\varphi k}$. The next Theorem is due to Katavalos–Paulsen [36, Section 3].

Theorem 1.57. *Let $\varphi \in \mathfrak{S}(X,Y)$. The map S_φ is a weak*-continuous, completely bounded, $\mathcal{D}_X$-$\mathcal{D}_Y$-bimodule map on $\mathcal{B}(L^2(X), L^2(Y))$. Conversely, if Φ is a weak*-continuous, completely bounded, $\mathcal{D}_X$-$\mathcal{D}_Y$-bimodule map on $\mathcal{B}(L^2(X), L^2(Y))$ then there exists a unique $\varphi \in \mathfrak{S}(X,Y)$ such that $\Phi = S_\varphi$.* □

Part of the following theorem was obtained by Grothendieck [25] when counting measure is used. For the proof see Todorov [56] and the comments of Pisier [46, Chapter 5].

Theorem 1.58. *Let $\varphi \in L^\infty(X \times Y)$. The following are equivalent:*

i. $\varphi \in \mathfrak{S}(X,Y)$ *and* $\|\varphi\|_{\mathfrak{S}} \le C$*;*

ii. there exist sequences $(a_k)_{k\in\mathbb{N}} \subseteq L^\infty(X)$ *and* $(b_k)_{k\in\mathbb{N}} \subseteq L^\infty(Y)$ *such that*

$$\operatorname*{ess\,sup}_{x\in X} \sum_{k=1}^{\infty} |a_k(x)|^2 \le C \text{ and } \operatorname*{ess\,sup}_{y\in Y} \sum_{k=1}^{\infty} |b_k(y)|^2 \le C,$$

and

$$\varphi(x,y) = \sum_{k=1}^{\infty} a_k(x) b_k(y)$$

for almost all $(x,y) \in X \times Y$*;*

iii. there exists a separable Hilbert space $\mathcal{H}$*, and measurable functions* $V : X \to \mathcal{H}$ *and* $W : Y \to \mathcal{H}$*, such that*

$$\operatorname*{ess\,sup}_{x\in X} \|V(x)\| \le \sqrt{C} \text{ and } \operatorname*{ess\,sup}_{y\in Y} \|W(y)\| \le \sqrt{C},$$

and

$$\varphi(x,y) = \langle V(x), W(y)\rangle$$

for almost all $(x,y) \in X \times Y$*.* □

1.4.2 Herz–Schur multipliers and transference

Let G be a locally compact group. The theory of Herz–Schur multipliers on G goes back to Herz's definition of the space $B_2(G)$ [30]. It was proved by Bożejko and Fendler [10], using a result of De Cannière and Haagerup [17] and unpublished work of Gilbert (see also Jolissaint [33]), that Herz's $B_2(G)$ is isometrically isomorphic to the space of completely bounded multipliers of the Fourier algebra of G, defined below. The overview here is drawn from the references mentioned above.

Definition 1.59. Let G be a locally compact group. A function $u : G \to \mathbb{C}$ is called a *multiplier* of $A(G)$ if $uv \in A(G)$ for every $v \in A(G)$, where $uv(t) := u(t)v(t)$ $(t \in G)$. We denote the space of multipliers of $A(G)$ by $\mathrm{M}A(G)$.

Since $A(G)$ is an ideal of $B(G)$ we have that $B(G) \subseteq \mathrm{M}A(G)$. It follows from the Closed Graph Theorem that if $u \in \mathrm{M}A(G)$ then the multiplication map

$$m_u : A(G) \to A(G);\ m_u(v) := uv, \quad v \in A(G), \tag{1.7}$$

is bounded. We endow $\mathrm{M}A(G)$ with the norm $\|u\|_{\mathrm{M}} := \|m_u\|$. As $A(G)$ is the predual of $\mathrm{vN}(G)$ it carries an operator space structure, so we can consider completely bounded maps on $A(G)$.

Definition 1.60. Let G be a locally compact group. A function $u : G \to \mathbb{C}$ is called a *completely bounded multiplier of* $A(G)$, or a *Herz–Schur multiplier of* G, if the multiplication map m_u of (1.7) is completely bounded. We denote the space of all Herz–Schur multipliers of G by $\mathrm{M}^{\mathrm{cb}}A(G)$, and give it the norm defined by $\|u\|_{\mathrm{M}^{\mathrm{cb}}} := \|m_u\|_{\mathrm{cb}}$.

If u is a Herz–Schur multiplier of G then by duality the map m_u gives rise to a completely bounded map on $\mathrm{vN}(G)$, which we denote by S_u. Indeed, if $v \in A(G)$ and $s \in G$ then

$$\left\langle S_u(\lambda_s^G), v\right\rangle = \left\langle \lambda_s^G, m_u(v)\right\rangle = \left\langle \lambda_s^G, uv\right\rangle = u(s)v(s) = \left\langle u(s)\lambda_s^G, v\right\rangle, \tag{1.8}$$

so that $S_u(\lambda_s^G) = u(s)\lambda_s^G$. Moreover, the restriction of S_u to $C_\lambda^*(G)$ is a completely bounded map from $C_\lambda^*(G)$ to itself, satisfying

$$S_u\big(\lambda^G(f)\big) = \lambda^G(uf), \quad f \in L^1(G). \tag{1.9}$$

Indeed, it follows from (1.8) that a multiplier u of $A(G)$ is automatically a bounded function on G, so the pointwise product uf belongs to $L^1(G)$ if f does. This argument implies a part of the following Theorem, essentially due to De Cannière–Haagerup [17, Proposition 1.2].

Theorem 1.61. *Let G be a locally compact group and $u : G \to \mathbb{C}$ a bounded, continuous function. The following are equivalent:*

i. $u \in \mathrm{M}A(G)$;

ii. there is a bounded, weak-continuous map T on $\mathrm{vN}(G)$ such that*

$$T(\lambda_s^G) = u(s)\lambda_s^G, \quad s \in G;$$

iii. there is a bounded linear map R on $C_\lambda^*(G)$ such that

$$R\big(\lambda^G(f)\big) = \lambda^G(uf), \quad f \in L^1(G);$$

iv. $uv \in B_\lambda(G)$ *for all* $v \in B_\lambda(G)$.

Moreover, $u \in \mathrm{M}^{\mathrm{cb}}A(G)$ if and only if the map T of condition (ii), equivalently the map R of condition (iii), is completely bounded. □

If G and Γ are locally compact groups and $u : G \to \mathbb{C}$ then we define $u^\Gamma : \Gamma \times G \to \mathbb{C}$; $u^\Gamma(\gamma, t) := u(t)$ $(t \in G,\ \gamma \in \Gamma)$. We have the following characterisation due to De Cannière–Haagerup [17, Theorem 1.6].

Theorem 1.62. *Let G be a locally compact group and $u \in \mathrm{M}A(G)$. The following are equivalent:*

i. u is a Herz–Schur multiplier of G;

ii. $u^\Gamma \in \mathrm{M}A(\Gamma \times G)$ *for every locally compact group* Γ;

iii. $u^{\mathrm{SU}(2)} \in \mathrm{M}A(\mathrm{SU}(2) \times G)$.

Moreover, if these conditions are satisfied then

$$\|u\|_{\mathrm{M}^{\mathrm{cb}}} = \sup_\Gamma \|u^\Gamma\|_{\mathrm{M}} = \|u^{\mathrm{SU}(2)}\|_{\mathrm{M}},$$

where the supremum is taken over all locally compact groups. □

The following characterisation of Herz–Schur multipliers, due to Bożejko–Fendler [10] and Jolissaint [33], will be useful.

Theorem 1.63. *Let G be a locally compact group and $u : G \to \mathbb{C}$. The following are equivalent:*

i. u is a Herz–Schur multiplier of G;

ii. there exist a Hilbert space $\mathcal{H}$, and bounded, continuous functions $V, W : G \to \mathcal{H}$, such that

$$u(ts^{-1}) = \langle V(s), W(t) \rangle, \quad s, t \in G.$$

Moreover, if the conditions hold then $\|u\|_{\mathrm{M^{cb}}} = \inf \|V\|_\infty \|W\|_\infty$, where the infimum is taken over all representations satisfying condition (ii). □

Let ρ^G denote the *right regular representation* of G on $L^2(G)$, given by

$$\rho^G : G \to \mathcal{B}(L^2(G));\ (\rho^G_t \xi)(s) := \Delta(t)^{\frac{1}{2}} \xi(st), \quad s, t \in G,\ \xi \in L^2(G),$$

and recall that $\operatorname{Ad} U$ denotes conjugation by U, *i.e.* $\operatorname{Ad} U(T) := UTU^*$. Let us now define the invariant Schur multipliers.

Definition 1.64. Let G be a locally compact group. A Schur multiplier $\varphi \in \mathfrak{S}(G, G)$ is called *invariant* if $\operatorname{Ad} \rho^G_r \circ S_\varphi = S_\varphi \circ \operatorname{Ad} \rho^G_r$ for all $r \in G$.

We summarise two characterisations of invariant Schur multipliers; the statements are drawn from Bożejko–Fendler [10], Herz [30], and Pisier [46, Theorem 6.4]; see also Todorov [56, Section 5].

Proposition 1.65. *Let G be a locally compact group and $\varphi : G \times G \to \mathbb{C}$. The following are equivalent:*

i. φ is an invariant Schur multiplier;

ii. S_φ *leaves* $\mathrm{vN}(G)$ *invariant;*

iii. for every $r \in G$ *the function* φ *is equal almost everywhere to the function*

$$\varphi_r : G \times G \to \mathbb{C};\ \varphi_r(s,t) := \varphi(sr, tr), \quad s, t \in G. \qquad \square$$

Let $u : G \to \mathbb{C}$ be a function. Define

$$N(u) : G \times G \to \mathbb{C};\ N(u)(s,t) := u(ts^{-1}), \quad s, t \in G. \tag{1.10}$$

Observe that $N(u)$ is measurable, or continuous, if u is so. A proof of the following Theorem is given by Jolissaint [33].

Theorem 1.66. *Let* G *be a locally compact group. The map* N *defined by (1.10) is an isometry from* $\mathrm{M}^{\mathrm{cb}}A(G)$ *to* $\mathfrak{S}(G,G)$*. Moreover,* N *maps onto the space of invariant Schur multipliers on* $G \times G$*.* $\square$

Note that the Fourier–Stieltjes algebra is contained in $\mathrm{M}^{\mathrm{cb}}A(G)$; it is natural to ask what other functions can be completely bounded multipliers of the Fourier algebra. It turns out that this question is related to amenability of the group G. Steenstrup [54] has obtained a characterisation of $\mathrm{M}^{\mathrm{cb}}A(G)$, for second-countable groups G, in terms of coefficients of representations which are not necessarily uniformly bounded.

1.5 Vector-valued integration

In what follows we will need to integrate functions which take values in a Banach space. In this section we collect together the necessary results and definitions. A thorough discussion of vector-valued integration for crossed products is given by Williams [60, Appendix B], which we use as a basis for this section.

1.5.1 General theory

Here we build up the standard theory, defining general integrals as limits of integrals of simple functions. Since the functions to be integrated may take values in a non-separable space, but the image of a sequence of simple functions is separable, we give a name to functions which satisfy this basic prerequisite.

Definition 1.67. Let X be a locally compact Hausdorff space and B a Banach space. We say $f : X \to B$ is *essentially separately-valued* on $S \subseteq X$ if there is a countable subset (equivalently a separable subspace) $D \subseteq B$ and a null set $N \subseteq S$ such that $f(x) \in \overline{D}$ for all $x \in S \setminus N$.

There are three notions of measurability, which we now define.

Definition 1.68. Let X be a locally compact Hausdorff space, μ a Radon measure on X, B a Banach space, and $f : X \to B$. We say:

- f is *strongly measurable* if:
    i. $f^{-1}(V)$ is measurable for every open set $V \subseteq B$, and
    ii. f is essentially separately-valued on every compact subset of X;
- f is *weakly measurable* if:
    i. $\phi \circ f : X \to \mathbb{C}$ is a measurable function for all $\phi \in B^*$, and
    ii. f is essentially separately-valued on every compact subset of X;
- f is *C-measurable* if for any compact subset $K \subseteq X$ and any $\epsilon > 0$ there is a compact subset $K' \subseteq K$ such that $\mu(K \setminus K') < \epsilon$ and the restriction $f|_{K'}$ is continuous.

It turns out that these three notions are equivalent [60, Lemma B.7, Proposition B.20]; for the Hilbert space connection see [60, Lemma I.23].

Lemma 1.69. *Let X be a locally compact Hausdorff space, B a Banach space, and $f : X \to B$. The following are equivalent:*

i. f is strongly measurable;

ii. f is weakly measurable;

iii. f is C-measurable.

If f satisfies any of these conditions we simply say f is measurable.

Moreover, if B is a Hilbert space, denoted by $\mathcal{H}$, then the above conditions are equivalent to

iv. for every $\xi \in \mathcal{H}$ the scalar-valued function $x \mapsto \langle f(x), \xi \rangle$ is measurable. □

Definition 1.70. Let X be a locally compact Hausdorff space, μ a Radon measure on X, and B a Banach space. We call a measurable function $f : X \to B$ *simple* if it takes only finitely many values $b_1, \ldots, b_n$ and $\mu(\{x \in X : f(x) = b_i\}) < \infty$ if $b_i \neq 0$.

Measurable functions can be characterised in terms of simple functions [60, Proposition B.24]

Proposition 1.71. *Let X be a locally compact Hausdorff space, B a Banach space, and $f : X \to B$. Then f is measurable if and only if, for each compact set $K \subseteq X$, there is a sequence of simple functions $(f_n)_{n \in \mathbb{N}}$ such that for almost all $x \in K$*

$$\|f_n(x)\| \leq \|f(x)\| \quad \textit{and} \quad f_n(x) \to f(x). \qquad \square$$

Definition 1.72. Let X be a locally compact Hausdorff space, μ a Radon measure on X, and B a Banach space. We say that a measurable function $f : X \to B$ is *integrable* if

$$\|f\|_1 := \int_X \|f(x)\| \, d\mu(x) < \infty.$$

The quantity $\|\cdot\|_1$ is called the L^1*-norm.* The collection of all integrable functions from X to B is denoted $\mathfrak{L}^1(X, B)$. The quotient space of integrable functions from X to B modulo almost-everywhere equivalence is a Banach space under $\|\cdot\|_1$ denoted $L^1(X, B)$.

If the measure μ is counting measure we write $\ell^1(X, B)$ in place of $L^1(X, B)$.

A useful fact is the density of compactly supported functions [60, Proposition B.33].

Proposition 1.73. *Let X be a locally compact Hausdorff space and B a Banach space. Then both the collection of simple functions and the space $C_c(X,B)$ are dense in $L^1(X,B)$.* □

Finally we can give the integral and its properties [60, Proposition B.34].

Proposition 1.74. *Let X be a locally compact Hausdorff space, μ a Radon measure on X, and B a Banach space. The map*

$$f \mapsto \int_X f(x)\, d\mu(x)$$

is a linear map satisfying

$$\left\| \int_X f(x)\, d\mu(x) \right\| \leq \|f\|_1, \quad f \in \mathfrak{L}^1(X,B).$$

The integral is characterised by

$$\phi\Big(\int_X f(x)\, d\mu(x)\Big) = \int_X \phi\big(f(x)\big)\, d\mu(x), \quad f \in \mathfrak{L}^1(X,B),\ \phi \in B^*.$$

Moreover, for any bounded linear map $T: B \to C$,

$$T\Big(\int_X f(x)\, d\mu(x)\Big) = \int_X T\big(f(x)\big)\, d\mu(x), \quad f \in \mathfrak{L}^1(X,B).$$ □

Proposition 1.74 in particular implies that if $f : X \to B$ is a simple function $f = \sum_{i=1}^n b_i \chi_{E_i}$, where $b_i \in B$ and each $E_i \subset X$ is of finite measure, then

$$\int_X f(x)\, d\mu(x) = \sum_{i=1}^n \mu(E_i) b_i.$$

Together with Proposition 1.73 this gives a concrete definition for the integral of a measurable function $f : X \to B$.

We will often need to work with product measures. If (X,μ) and (Y,ν) are measure spaces with μ and ν Radon measures we want to define a product Radon measure. The function

$$J : C_c(X \times Y) \to \mathbb{C};\ J(f) := \int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x) = \int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(y)$$

is a positive linear functional on $C_c(X \times Y)$ (the two integrals are the same by Fubini's Theorem [13, Proposition 7.6.4]). We define the *product of* μ *and* ν to be the Radon measure associated to the functional J by the Riesz Representation Theorem, Theorem 1.35. The following Theorem [60, Theorem B.41], essentially a Fubini Theorem for vector-valued functions, will be useful.

Theorem 1.75. *Let* (X, μ) *and* (Y, ν) *be locally compact Hausdorff spaces equipped with Radon measures and* B *a Banach space. If* $f \in \mathfrak{L}^1(X \times Y, B)$ *then the following hold:*

i. for almost all $x \in X$ *the function* $y \mapsto f(x, y)$ *belongs to* $\mathfrak{L}^1(Y, B)$*;*

ii. for almost all $y \in Y$ *the function* $x \mapsto f(x, y)$ *belongs to* $\mathfrak{L}^1(X, B)$*;*

iii. the function

$$y \mapsto \int_X f(x,y)\, d\mu(x)$$

is defined almost everywhere and defines an equivalence class in $L^1(Y, B)$*;*

iv. the function

$$x \mapsto \int_Y f(x,y)\, d\nu(y)$$

is defined almost everywhere and defines an equivalence class in $L^1(X, B)$*;*

v. the iterated integrals

$$\int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x) \quad \textit{and} \quad \int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(y)$$

are both equal to

$$\int_{X \times Y} f(x,y)\, d(\mu \times \nu)(x,y).$$

□

Definition 1.76. Let (X, μ) be a measure space and E and F be Banach spaces. A function $\phi : X \to \mathcal{B}(E, F)$ will be called *pointwise-measurable* if, for every $a \in E$, the function $x \mapsto \phi(x)(a)$ is a measurable function from X to F.

We will require an improvement of a result given by Takesaki [55, Lemma IV.7.5].

Lemma 1.77. *Let X be a locally compact Hausdorff space equipped with a Radon measure and E and F Banach spaces. Suppose that $\phi : X \to \mathcal{B}(E, F)$ is a pointwise-measurable function. Then, for any measurable function $\xi : X \to E$, the function*

$$X \to F;\ x \mapsto \phi(x)\big(\xi(x)\big), \quad x \in X,$$

is also measurable.

Proof. Let $\zeta : X \to F$ denote the function in question. Let $K \subseteq X$ be a compact set. By Proposition 1.71 there is a sequence of simple functions $(\xi_n)_{n\in\mathbb{N}}$ such that $\xi_n(x) \to \xi(x)$ for almost all $x \in K$. It is clear from the definition of pointwise-measurability that the functions

$$\zeta_n : X \to F;\ x \mapsto \phi(x)\big(\xi_n(x)\big), \quad x \in X,$$

are measurable for all $n \in \mathbb{N}$. Since we have $\zeta_n|_K \to \zeta|_K$ almost everywhere it follows from Williams [60, Lemma B.11] that ζ is measurable. □

1.5.2 Square-integrable functions

We will often need to work with vector-valued L^2 spaces. Here we collect the necessary definitions and facts about such spaces for later use. The discussion is based on Takesaki [55, Section IV.7] and Williams [60, Section I.4]. Note that Takesaki's definition of a measurable function [55, Definition IV.7.1] agrees with our definition of C-measurability.

Recall that X is a locally compact Hausdorff space, equipped with a Radon measure μ, and B a Banach space.

Definition 1.78. Let X be a locally compact Hausdorff space equipped with a Radon measure μ and B a Banach space. Let $f : X \to B$ be a measurable function and define

$$\|f\|_2 := \left(\int_X \|f(x)\|^2 \, d\mu(x)\right)^{\frac{1}{2}}. \tag{1.11}$$

The functions f such that $\|f\|_2$ is well-defined (*i.e.* $x \mapsto \|f(x)\|^2$ is integrable) and finite are called *square-integrable*, and the set of such functions is denoted by $\mathfrak{L}^2(X, B)$. The quotient of $\mathfrak{L}^2(X, B)$ by those functions with $\|\cdot\|_2$ equal to zero is a normed space under $\|\cdot\|_2$, the Banach space obtained by completing this last is denoted by $L^2(X, B)$.

Identify the algebraic tensor product $C_c(X) \odot B$ with a linear space of B-valued functions by

$$\sum_{i=1}^{n} f_i \otimes b_i \leftrightarrow \sum_{i=1}^{n} f_i(\cdot) b_i, \quad f_i \in C_c(X),\ b_i \in B. \tag{1.12}$$

This identifies $C_c(X) \odot B$ with the space of continuous B-valued functions on X with compact support and finite-dimensional range. Note that (1.11) defines a seminorm on such functions. The following result is given by Takesaki [55, Proposition IV.7.4].

Proposition 1.79. *Let X be a locally compact Hausdorff space equipped with a Radon measure μ and B a Banach space. The space $L^2(X, B)$ is canonically identified with the completion under $\|\cdot\|_2$ of $(C_c(X) \odot B)/N$, where $N = \{f \in C_c(X) \odot B : \|f\|_2 = 0\}$.* □

In particular, using this result and the identification above, we may treat finite sums $\sum_{i=1}^{n} f_i \otimes b_i \in C_c(X) \odot B$ as being dense in $L^2(X, B)$.

An important case of the theory so far is when B is a Hilbert space, which we now examine further. The next result is given by Williams [60, Lemma I.19].

Lemma 1.80. *Let X be a locally compact, Hausdorff space equipped with a Radon measure μ and $\mathcal{H}$ a Hilbert space. Let $\xi, \eta \in \mathfrak{L}^2(X, \mathcal{H})$. Then the function on X given by $x \mapsto \langle \xi(x), \eta(x) \rangle$ is integrable, and*

$$\langle \xi, \eta \rangle := \int_X \langle \xi(x), \eta(x) \rangle \, d\mu(x)$$

defines an inner product on $L^2(X, \mathcal{H})$. □

Observe that for $\xi \in L^2(X, \mathcal{H})$

$$\|\xi\|_2^2 = \int_X \|\xi(x)\|^2 \, d\mu(x) = \int_X \langle \xi(x), \xi(x) \rangle \, d\mu(x) = \langle \xi, \xi \rangle \,,$$

so that the norm on $L^2(X, \mathcal{H})$ defined in (1.11) is the inner product norm. It follows that $L^2(X, \mathcal{H})$ is a Hilbert space as it is complete in the inner product norm [60, Lemma I.20].

The results so far imply that $C_c(X, \mathcal{H})$ is dense in $L^2(X, \mathcal{H})$. Observe that since $C_c(X)$ is dense in $L^2(X)$ we have, from Proposition 1.79, that the identification of (1.12) extends to a unitary equivalence of $L^2(X, \mathcal{H})$ and the Hilbert space tensor product $L^2(X) \otimes \mathcal{H}$. I will switch freely between these two spaces, without comment, throughout this thesis.

Occasionally we will be considering $L^2(X, B)$ when X is a discrete space equipped with counting measure. In this case I will write $\ell^2(X, B)$.

1.5.3 Essentially bounded functions

Recall once again that X is a locally compact, Hausdorff space equipped with a Radon measure μ. Here we define vector-valued versions of L^∞ functions, and state an important result which will be needed several times. The main reference for this section is Takesaki [55, Section IV.7].

Definition 1.81. Let X be a locally compact, Hausdorff space equipped with a Radon measure μ and $\mathcal{H}$ a separable Hilbert space. Let $L^\infty(X, \mathcal{B}(\mathcal{H}))$ denote the collection of all essentially bounded, measurable functions $f : X \to \mathcal{B}(\mathcal{H})$ endowed with the norm

$$\|f\|_\infty := \operatorname*{ess\,sup}_{x \in X} \|f(x)\|.$$

This space is called the space of *essentially bounded functions* from X to $\mathcal{B}(\mathcal{H})$.

Associated to each element $f \in L^\infty(X, \mathcal{B}(\mathcal{H}))$ is a bounded operator on $L^2(X, \mathcal{H})$ given by

$$M_f : L^2(X, \mathcal{H}) \to L^2(X, \mathcal{H});\ (M_f \xi)(x) := f(x)\big(\xi(x)\big), \quad \xi \in L^2(X, \mathcal{H}),\ x \in X.$$

as in (1.3). In particular, for any (separable) Hilbert space $\mathcal{H}$, there is an obvious inclusion of $\mathcal{D}_X$ into $L^\infty(X, \mathcal{B}(\mathcal{H}))$, so that each $f \in L^\infty(X)$ gives rise to a bounded operator

M_f acting on $L^2(X, \mathcal{H})$. The space of all such bounded operators on $L^2(X, \mathcal{H})$ arising from elements of $L^\infty(X)$ will again be denoted by $\mathcal{D}_X$. Observe that the unitary which identifies $L^2(X, \mathcal{H})$ with $L^2(X) \otimes \mathcal{H}$ implements a unitary equivalence of $L^\infty(X, \mathcal{B}(\mathcal{H}))$ and $\mathcal{D}_X \overline{\otimes} \mathcal{B}(\mathcal{H})$.

The following result, given by Takesaki [55, Theorem IV.7.10], characterises those elements of $\mathcal{B}(L^2(X, \mathcal{H}))$ which are of the form M_f for some $f \in L^\infty(X, \mathcal{B}(\mathcal{H}))$.

Theorem 1.82. *Let X be a locally compact Hausdorff space equipped with a Radon measure μ and $\mathcal{H}$ a separable Hilbert space. Let $T \in \mathcal{B}(L^2(X, \mathcal{H}))$. The following are equivalent:*

i. T is an operator of the form M_f for some $f \in L^\infty(X, \mathcal{B}(\mathcal{H}))$;

ii. T commutes with $\mathcal{D}_X$. □

Remark 1.83. In applications of the material in this section we will be considering operators in $\mathcal{B}(L^2(X, \mathcal{H}), L^2(X, \mathcal{L}))$, where $\mathcal{H}$ and $\mathcal{L}$ are two separable Hilbert spaces. Since we have so far only considered the case $\mathcal{H} = \mathcal{L}$ we observe that Theorem 1.82 applies in the more general situation. Suppose $T \in \mathcal{B}(L^2(X, \mathcal{H}), L^2(X, \mathcal{L}))$ commutes with $\mathcal{D}_X$. Identifying $\mathcal{B}(L^2(X, \mathcal{H}), L^2(X, \mathcal{L}))$ with a subspace of $\mathcal{B}(L^2(X, \mathcal{H}) \oplus L^2(X, \mathcal{L}))$, and $L^2(X, \mathcal{H}) \oplus L^2(X, \mathcal{L})$ with $L^2(X) \otimes (\mathcal{H} \oplus \mathcal{L})$ using the distributivity of the tensor product, and applying Theorem 1.82 we obtain $f \in L^\infty(X, \mathcal{B}(\mathcal{H} \oplus \mathcal{L}))$ such that

$$T\xi(x) = f(x)\big(\xi(x)\big), \quad x \in X, \ \xi \in L^2(X, \mathcal{H}).$$

It follows that $f(x) \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ for all $x \in X$. On the other hand, it is clear that if T is an operator of the form M_f for some $f \in L^\infty(X, \mathcal{B}(\mathcal{H}, \mathcal{L}))$ then T commutes with $\mathcal{D}_X$.

1.6 Dynamical systems and crossed products

In Section 1.4.2 Herz–Schur multipliers were defined on a group, and Theorem 1.61 shows how Herz–Schur multipliers can be used to study the reduced group C^*-algebra, or group von Neumann algebra. A primary goal of this thesis is to develop similar tools for the study of group actions and their associated operator algebras. In this section we give the necessary background. *Dynamical system* has a broad meaning, coming from traditional descriptions of dynamics through differential equations; we focus on the abstract C^*-algebra case, which amounts to a group acting on a C^*-algebra by automorphisms. More information on classical dynamical systems, and how they motivate the C^*-algebraic notion defined below, can be found in Raeburn [47].

This section is based on Willams [60, Chapter 2] and Pedersen [44, Section 7.6].

Definition 1.84. Let A be a C^*-algebra, G a locally compact group, and $\alpha : G \to \operatorname{Aut}(A)$ a point-norm continuous homomorphism (*i.e.* for all $a \in A$ the map $s \mapsto \alpha_s(a)$ is continuous from G to A). The triple (A, G, α) is called a *C^*-dynamical system.*

Since we are studying dynamical systems from a C^*-algebra perspective it is natural to represent on a Hilbert space.

Definition 1.85. Let (A, G, α) be a C^*-dynamical system. A *covariant representation* of (A, G, α) is a pair (ρ, τ) of representations of A and G respectively on a Hilbert space $\mathcal{H}$, such that

$$\rho\big(\alpha_s(a)\big) = \tau_s \rho(a) \tau_s^*, \quad a \in A,\ s \in G. \tag{1.13}$$

The following examples are given by Williams [60, Example 2.11].

Example 1.86. The covariant representations of the system $(A, \{e\}, \mathrm{id})$ obviously correspond to representations of A. Similarly, covariant representations of $(\mathbb{C}, G, \mathrm{id})$ correspond to unitary representations of G.

In constructing the reduced crossed product below we will show that even in non-trivial cases a covariant representation always exists. The collection of all covariant pairs of representations of the system (A, G, α) will be denoted $\Sigma(A, G, \alpha)$, or simply Σ when the C^*-dynamical system is clear from context.

Let (A, G, α) be a C^*-dynamical system. As in Definition 1.72 we can form the Banach space $L^1(G, A)$. We now define the convolution product and involution on this Banach space, so that it becomes a Banach $*$-algebra. We then construct the crossed product by building a C^*-algebra using covariant representations of (A, G, α) and $L^1(G, A)$.

Recall from Proposition 1.73 that $C_c(G,A)$ is dense in $L^1(G,A)$. We first define convolution and involution on $C_c(G,A)$, which extend to $L^1(G,A)$ by continuity, then state a result which gives a representative of the convolution product of two elements of $L^1(G,A)$ and the involution of an element of $L^1(G,A)$.

Definition 1.87. Let (A,G,α) be a C^*-dynamical system and $f,g \in C_c(G,A)$. Then

$$f * g(s) := \int_G f(r)\alpha_r\big(g(r^{-1}s)\big)\,dr, \quad s \in G, \tag{1.14}$$

defines an element $f * g \in C_c(G,A)$ called the *convolution product* of f and g. The formula

$$f^*(s) := \Delta(s^{-1})\alpha_s\big(f(s^{-1})^*\big), \quad s \in G, \tag{1.15}$$

defines an element $f^* \in C_c(G,A)$ called the *involution* of f. With these operations $C_c(G,A)$ becomes a $*$-algebra.

The result below is from Williams [60, Proposition B.42].

Proposition 1.88. *Let (A,G,α) be a C^*-dynamical system and $f,g \in \mathfrak{L}^1(G,A)$. Then there is a null set $M \subset G$ such that $s \notin M$ implies $r \mapsto f(r)\alpha_r(g(r^{-1}s))$ is in $\mathfrak{L}^1(G,A)$. Moreover, if we define*

$$\kappa(s) := \int_G f(r)\alpha_r\big(g(r^{-1}s)\big)\,dr, \quad s \notin M,$$

and $\kappa(s) := 0$ if $s \in M$, then κ is an element of $\mathfrak{L}^1(G,A)$ representing the convolution product $f * g$ of f and g (defined as the extension of (1.14) to $L^1(G,A)$).

Also

$$\iota(s) := \Delta(s^{-1})\alpha_s\big(f(s^{-1})^*\big), \quad s \in G,$$

defines an element $\iota \in \mathfrak{L}^1(G,A)$ representing the involution of f (defined as the extension of (1.15) to $L^1(G,A)$). □

Now we investigate how covariant representations of a C^*-dynamical system give rise to representations of the associated $*$-algebra. The following result is from Williams [60, Proposition 2.23].

Proposition 1.89. *Let (A, G, α) be a C^*-dynamical system and (ρ, τ) a covariant representation of (A, G, α) on a Hilbert space $\mathcal{H}$. Then*

$$\rho \rtimes \tau(f) := \int_G \rho\big(f(s)\big)\tau_s \, ds, \quad f \in C_c(G, A),$$

defines a representation of $C_c(G, A)$ called the integrated form *of (ρ, τ), which is L^1-norm decreasing.* □

Since the integrated form of a covariant representation is L^1-norm decreasing it extends to a representation of $L^1(G, A)$; this fact will be used to define Herz–Schur multipliers of a C^*-dynamical system in Chapter 3.

Definition 1.90. Let (A, G, α) be a C^*-dynamical system. Define

$$\|f\|_\Sigma := \sup \|\rho \rtimes \tau(f)\|, \quad f \in C_c(G, A),$$

where the supremum is taken over all covariant pairs in Σ. Then $\|\cdot\|_\Sigma$ is a norm on $C_c(G, A)$ called the *universal norm*. We define the *crossed product* associated to the system (A, G, α) to be the completion of $C_c(G, A)$ under $\|\cdot\|_\Sigma$; denote this C^*-algebra by $A \rtimes_\alpha G$.

The following Proposition [60, Proposition 2.40] is the C^*-dynamical system analogue of the fact that characters on an abelian group are in one-to-one correspondence with $\mathbb{C}$-valued homomorphisms (*i.e.* one-dimensional representations) on the associated L^1 algebra, or the fact that unitary representations of a locally compact group are in one-to-one correspondence with representations of the associated group C^*-algebra.

Proposition 1.91. *Let (A, G, α) be a C^*-dynamical system. The map sending a covariant pair (ρ, τ) to its integrated form $\rho \rtimes \tau$ is a one-to-one correspondence between non-degenerate covariant representations of (A, G, α) and non-degenerate representations of $A \rtimes_\alpha G$. This correspondence preserves direct sums, irreducibility, and equivalence.* □

As with group C^*-algebras there is a reduced C^*-algebra associated to a C^*-dynamical system (A, G, α), related to the left regular representation of G. In order to define it we suppose $A \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then we define a new representation $\pi : A \to \mathcal{B}(L^2(G) \otimes \mathcal{H})$ by

$$\big(\pi(a)\xi\big)(s) := \alpha_{s^{-1}}(a)\big(\xi(s)\big), \quad a \in A,\ s \in G,\ \xi \in L^2(G, \mathcal{H}),$$

and a representation $\lambda : G \to \mathcal{U}(L^2(G) \otimes \mathcal{H})$ by

$$(\lambda_t \xi)(s) := \xi(t^{-1}s), \quad s, t \in G,\ \xi \in L^2(G, \mathcal{H}).$$

Note that $\lambda = \lambda^G \otimes I_{\mathcal{H}}$, where λ^G is the left regular representation of G on $L^2(G)$. For $a \in A,\ s, t \in G$ and $\xi \in L^2(G) \otimes \mathcal{H}$ we have

$$\begin{aligned} \lambda_t \pi(a) \lambda_t^*(\xi)(s) &= \pi(a)(\lambda_t^* \xi)(t^{-1}s) \\ &= \alpha_{s^{-1}t}(a)(\lambda_t^* \xi)(t^{-1}s) \\ &= \alpha_{s^{-1}t}(a)\big(\xi(s)\big) \\ &= \pi\big(\alpha_t(a)\big)\big(\xi(s)\big), \end{aligned} \tag{1.16}$$

so that (π, λ) is a covariant pair.

Definition 1.92. Let (A, G, α) be a C^*-dynamical system. The *reduced crossed product* associated to (A, G, α) is defined to be the completion of $C_c(G, A)$ in the norm

$$\|f\|_{(\pi,\lambda)} := \|\pi \rtimes \lambda(f)\|.$$

The resulting C^*-algebra is denoted $A \rtimes_{\alpha,r} G$.

The norm subscript will be omitted when it is clear from context. The next result, from Pedersen [44, Theorem 7.7.5], implies that we can construct the reduced crossed product beginning with any faithful representation of A.

Theorem 1.93. *Let (A, G, α) be a C^*-dynamical system and $(\theta, \mathcal{H}_\theta)$ a faithful representation of A. Let $\pi^\theta : A \to \mathcal{B}(L^2(G) \otimes \mathcal{H}_\theta)$ be given by*

$$\big(\pi^\theta(a)\xi\big)(s) := \theta\big(\alpha_{s^{-1}}(a)\big)\big(\xi(s)\big), \quad a \in A,\ s \in G,\ \xi \in L^2(G, \mathcal{H}_\theta),$$

and $\lambda^\theta : G \to \mathcal{U}(L^2(G) \otimes \mathcal{H}_\theta)$ *by*

$$(\lambda_t^\theta \xi)(s) := \xi(t^{-1}s), \quad s, t \in G,\ \xi \in L^2(G, \mathcal{H}_\theta).$$

Then $\pi^\theta \rtimes \lambda^\theta$ *is a faithful representation of* $A \rtimes_{\alpha,r} G$ *on* $L^2(G, \mathcal{H}_\theta)$. □

Exactly as in (1.16) we find that $(\pi^\theta, \lambda^\theta)$ is a covariant representation of (A, G, α). We let $A \rtimes_{\alpha,\theta} G := (\pi^\theta \rtimes \lambda^\theta)(A \rtimes_\alpha G)$. It follows from Theorem 1.93 that $A \rtimes_{\alpha,\theta} G = \overline{(\pi^\theta \rtimes \lambda^\theta)(L^1(G, A))}$ is isomorphic to $A \rtimes_{\alpha,r} G$.

We will occasionally need to work with the weak* closure of $A \rtimes_{\alpha,\theta} G$, which will be denoted $A \rtimes_{\alpha,\theta}^{\mathrm{w}^*} G$.

Chapter 2

Schur multipliers

In this chapter we introduce a vector-valued version of Schur multipliers. More specifically, we consider integral operators with kernels taking values in a C^*-algebra, which play the role of the Hilbert–Schmidt operators in Section 1.4.1, and define a general multiplier acting on such integral operators. We give a characterisation of such multipliers, similar to Theorem 1.58, and also show how they can be represented using Hilbert C^*-modules.

2.1 Vector-valued Hilbert–Schmidt operators

We begin by developing integral operators with vector-valued kernels. In this section (X, μ) and (Y, ν) will be standard measure spaces and $\mathcal{H}$ a separable Hilbert space; we will always assume that the topological space underlying a standard measure space is locally compact. Note that X and Y are separable and metrizable by definition, so therefore (by *e.g.* Cohn [13, D.32]) they are second-countable; since the underlying topological spaces are assumed to be locally compact it follows from Proposition 1.3 that X and Y are σ-compact, and therefore σ-finite since Radon measures take finite values on compact sets. The fact that the measure spaces (X, μ) and (Y, ν) are σ-finite will be used later.

If $k \in L^2(Y \times X, \mathcal{B}(\mathcal{H}))$ and $\xi \in L^2(X, \mathcal{H})$ then we claim that the function $(y,x) \mapsto k(y,x)(\xi(x))$ is a measurable function from $Y \times X$ to $\mathcal{H}$. We show that this function is C-measurable. First assume that X and Y are compact, and let $\epsilon > 0$. Then, since Y is compact and $x \mapsto \xi(x)$ is measurable, $(y,x) \mapsto \xi(x)$ is measurable as a function on $Y \times X$. Using the C-measurability of k and ξ we may find compact sets $L \subseteq Y \times X$ and $K \subseteq X$ such that $k|_L$ and $\xi|_K$ are continuous, and $(\mu \times \nu)(Y \times X \setminus L) < \epsilon$. It follows that the restriction of the function $(y,x) \mapsto k(y,x)(\xi(x))$ to the compact set $L \cap (Y \times K)$ is continuous and $(\mu \times \nu)(Y \times X \setminus L \cap (Y \times K)) < \epsilon$. Thus the function is C-measurable when X and Y are compact. Now if X and Y are not compact then, using the σ-compactness of (X,μ) and (Y,ν), find increasing sequences $(E_n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$ of compact subsets of X and Y respectively, satisfying $\cup_{n\in\mathbb{N}} E_n = X$ and $\cup_{n\in\mathbb{N}} F_n = Y$. Clearly $(F_n \times E_n)_{n\in\mathbb{N}}$ is an increasing sequence of compact subsets of $Y \times X$ with $\cup_{n\in\mathbb{N}}(F_n \times E_n) = Y \times X$. The argument so far shows that the restriction of the function in question to each $F_n \times E_n$ is measurable; it follows that the function is measurable on $\cup_{n\in\mathbb{N}}(F_n \times E_n) = Y \times X$. This proves the claim. Also note that for any $y \in Y$ the map $x \mapsto k(y,x)(\xi(x))$ is measurable, by the definition of (weak) measurability and the corresponding scalar-valued result [13, Lemma 5.1.2(b)]. Finally, we have

$$\int_X \|k(y,x)\big(\xi(x)\big)\|\, d\mu(x) \le \int_X \|k(y,x)\|\|\xi(x)\|\, d\mu(x) \le \|\xi\|_2 \left(\int_X \|k(y,x)\|^2\, d\mu(x)\right)^{\frac{1}{2}}. \tag{2.1}$$

Lemma 2.1. *Let $k \in L^2(Y \times X, \mathcal{B}(\mathcal{H}))$ and $\xi \in L^2(X, \mathcal{H})$. Then the function*

$$T_k\xi : Y \to \mathcal{H};\ (T_k\xi)(y) := \int_X k(y,x)\big(\xi(x)\big)\, d\mu(x), \quad y \in Y, \tag{2.2}$$

is measurable and defines a bounded linear operator $T_k : L^2(X,\mathcal{H}) \to L^2(Y,\mathcal{H})$ with $\|T_k\| \le \|k\|_2$. Moreover, $T_k = 0$ if and only if $k = 0$ almost everywhere.

Proof. It follows from (2.1) that the integral in (2.2) is well-defined. To show that $T_k\xi$ is measurable take an element $\eta \in \mathcal{H}$; since $(y,x) \mapsto k(y,x)(\xi(x))$ is measurable we have, by Fubini's Theorem [13, Theorem 5.2.2], Proposition 1.74, and Lemma 1.69, that

$$y \mapsto \int_X \langle k(y,x)\big(\xi(x)\big), \eta\rangle \, d\mu(x) = \left\langle \int_X k(y,x)\big(\xi(x)\big)\, d\mu(x), \eta \right\rangle$$

is measurable; by Lemma 1.69 this means that $T_k\xi$ is measurable.

The calculation below shows that $T_k\xi$ is square-integrable, so $T_k\xi \in L^2(Y, \mathcal{H})$;

$$\begin{aligned}
\|T_k\xi\|_2^2 &= \int_Y \|(T_k\xi)(y)\|^2 \, d\nu(y) \\
&= \int_{Y\times X} \|k(y,x)\big(\xi(x)\big)\|^2 \, d(\nu\times\mu)(y,x) \\
&\le \|\xi\|_2^2 \int_{Y\times X} \|k(y,x)\|^2 \, d(\nu\times\mu)(y,x) \\
&= \|k\|_2^2 \|\xi\|_2^2.
\end{aligned}$$

It also follows that T_k is bounded with norm at most $\|k\|_2$.

It is clear that if $k = 0$ almost everywhere then $T_k = 0$. Conversely, suppose $T_k = 0$ and choose a countable dense subset $\{e_i\}_{i\in\mathbb{N}}$ of $\mathcal{H}$. For any $\xi \in L^2(X)$, $\eta \in L^2(Y)$ we have

$$\int_{X\times Y} \langle k(y,x)e_i, e_j\rangle \, \xi(x)\overline{\eta(y)} \, d(\mu\times\nu)(x,y) = \langle T_k(\xi\otimes e_i), \eta\otimes e_j\rangle = 0,$$

which implies that $\langle k(y,x)e_i, e_j\rangle = 0$ almost everywhere, for all $i, j \in \mathbb{N}$. Since $k(y,x)$ is a bounded operator we have $k(y,x) = 0$ for almost all $(y,x) \in Y \times X$. $\square$

As in the classical case, Theorem 1.55, a function $k \in L^2(Y\times X, A)$ will be called a *kernel* and the associated map T_k the *integral operator with kernel k*.

Definition 2.2. Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a C^*-algebra. We define

$$\mathcal{S}_2(X, Y; A) := \{T_k : k \in L^2(Y\times X, A)\}.$$

When $A = \mathbb{C}$ we simply write $\mathcal{S}_2(X, Y)$, which is the space of Hilbert–Schmidt operators from $L^2(X)$ to $L^2(Y)$, defined in Definition 1.54, by Theorem 1.55. If $h \in L^2(Y \times X)$

and $a \in A$ then it is obvious that

$$T_{h\otimes a} = T_h \otimes a. \tag{2.3}$$

Since $\mathcal{S}_2(X,Y)\odot A$ is norm-dense in $\mathcal{K}(L^2(X),L^2(Y))\otimes_{\min} A$ it follows that $\mathcal{S}_2(X,Y;A)$ is norm-dense in $\mathcal{K}(L^2(X),L^2(Y))\otimes_{\min} A$. We equip $\mathcal{S}_2(X,Y;A)$ with the operator space structure arising from this inclusion.

2.2 Schur multipliers

From this point we fix a non-degenerate, separable, C^*-algebra $A \subseteq \mathcal{B}(\mathcal{H})$. Let $\varphi : X \times Y \to \mathcal{CB}(A,\mathcal{B}(\mathcal{H}))$ be a bounded, pointwise-measurable function. For $k \in L^2(Y\times X, A)$ define

$$\varphi \cdot k : Y \times X \to \mathcal{B}(\mathcal{H});\ (\varphi\cdot k)(y,x) := \varphi(x,y)\big(k(y,x)\big), \quad (y,x) \in Y \times X.$$

The function $\varphi\cdot k$ is measurable by Lemma 1.77. If $\|\varphi\|_\infty = \sup_{(x,y)\in X\times Y}\|\varphi(x,y)\|$ then

$$\begin{aligned}\|\varphi\cdot k\|_2^2 &= \int_{Y\times X}\|\varphi\cdot k(y,x)\|^2\, d(\nu\times\mu)(y,x)\\ &\le \int_{Y\times X}\|\varphi(x,y)\|^2\|k(y,x)\|^2\, d(\nu\times\mu)(y,x)\\ &\le \|\varphi\|_\infty^2\|k\|_2^2;\end{aligned}$$

thus $\varphi\cdot k \in L^2(Y\times X,\mathcal{B}(\mathcal{H}))$ and $\|\varphi\cdot k\|_2 \le \|\varphi\|_\infty\|k\|_2$. Let $S_\varphi : \mathcal{S}_2(X,Y;A) \to \mathcal{S}_2(X,Y;\mathcal{B}(\mathcal{H}))$ be the linear map defined by

$$S_\varphi(T_k) := T_{\varphi\cdot k}, \quad k \in L^2(Y\times X, A). \tag{2.4}$$

By Lemma 2.1, and the fact that $\varphi\cdot k \in L^2(Y\times X,\mathcal{B}(\mathcal{H}))$, the map S_φ is well-defined.

We now define our generalised Schur multipliers, motivated by the similarity of (2.4) and (1.6).

Definition 2.3. Let $\varphi : X \times Y \to \mathcal{CB}(A, \mathcal{B}(\mathcal{H}))$ be a bounded, pointwise-measurable function. We call φ a *Schur A-multiplier* if the map S_φ of (2.4) is completely bounded. We let $\mathfrak{S}(X, Y; A, \mathcal{B}(\mathcal{H}))$ denote the space of all Schur A-multipliers from $X \times Y$ to $\mathcal{CB}(A, \mathcal{B}(\mathcal{H}))$ and endow it with the norm

$$\|\varphi\|_{\mathfrak{S}} := \|S_\varphi\|_{\mathrm{cb}}.$$

It follows from Lemma 2.1 that if $S_\varphi = 0$ then $\varphi = 0$ almost everywhere, so $\|\cdot\|_{\mathfrak{S}}$ is indeed a norm on $\mathfrak{S}(X, Y; A, \mathcal{B}(\mathcal{H}))$.

Since, for any C^*-algebra B, $\mathcal{S}_2(X, Y; B)$ is norm-dense in $\mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} B$ a bounded, pointwise-measurable function $\varphi : X \times Y \to \mathcal{CB}(A, \mathcal{B}(\mathcal{H}))$ is a Schur A-multiplier if and only if S_φ has an extension to a completely bounded map from $\mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} A$ to $\mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} \mathcal{B}(\mathcal{H})$; such an extension will be denoted by the same symbol S_φ.

Remark 2.4. Suppose that $A = \mathbb{C}$ in Definition 2.3. Identifying $\mathcal{CB}(\mathbb{C})$ with $\mathbb{C}$ we have that $\varphi : X \times Y \to \mathbb{C}$ is a Schur $\mathbb{C}$-multiplier if and only if the map $T_k \mapsto T_{\varphi k}$ is completely bounded for all $k \in L^2(Y \times X)$; moreover S_φ is a $\mathcal{D}_X$-$\mathcal{D}_Y$-bimodule map (*e.g.* by Lemma 2.7 below). It follows from Theorem 1.57 and equation (1.6) that Schur $\mathbb{C}$-multipliers are the classical measurable Schur multipliers described in Subsection 1.4.1.

Definition 2.5. The space of all Schur A-multipliers of the form $\varphi : X \times Y \to \mathcal{CB}(A)$ will be denoted by $\mathfrak{S}_0(X, Y; A)$.

At first glance the space $\mathcal{S}_2(X, Y; A)$, and therefore the collection of Schur A-multipliers (which are defined by their action on $\mathcal{S}_2(X, Y; A)$), appears to depend on the representation of A. It may be convenient to work with $\mathfrak{S}_0(X, Y; A)$, since this space is independent of the faithful representation of A, as the following result shows.

Proposition 2.6. *Let $(\theta, \mathcal{H}_\theta)$ be a faithful representation of A on a separable Hilbert space. A bounded, pointwise-measurable map $\varphi : X \times Y \to \mathcal{CB}(A)$ is a Schur A-multiplier if and only if the bounded, pointwise-measurable map*

$$\varphi^\theta : X \times Y \to \mathcal{CB}(\theta(A));\ \varphi^\theta(x, y)\big(\theta(a)\big) := \theta\big(\varphi(x, y)(a)\big), \quad x \in X,\ y \in Y,\ a \in A,$$

is a Schur $\theta(A)$-multiplier. Moreover, $\|\varphi\|_{\mathfrak{S}} = \|\varphi^\theta\|_{\mathfrak{S}}$.

Proof. It is clear that φ^θ is well-defined, and bounded and pointwise-measurable if and only if φ is so. The map

$$\mathrm{id} \otimes \theta : \mathcal{K}(L^2(X), L^2(Y)) \odot A \to \mathcal{K}(L^2(X), L^2(Y)) \odot \theta(A)$$

given by

$$(\mathrm{id} \otimes \theta)(T \otimes a) := T \otimes \theta(a), \quad T \in \mathcal{K}(L^2(X), L^2(Y)),\ a \in A,$$

extends to a complete isometry

$$\mathrm{id} \otimes \theta : \mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} A \to \mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} \theta(A)$$

(see *e.g.* Effros–Ruan [22, Proposition 8.1.5]). Let $k \in L^2(Y \times X, A)$; then, since

$$\begin{aligned}\int_{Y\times X} \|\theta\big(k(y,x)\big)\|^2 \, d(\nu \times \mu)(y,x) &\leq \int_{Y\times X} \|\theta\|^2 \|k(y,x)\|^2 \, d(\nu \times \mu)(y,x) \\ &= \|k\|_2^2,\end{aligned}$$

we have $\theta \circ k \in L^2(Y \times X, \theta(A))$. We show that

$$T_{\theta\circ k} = (\mathrm{id} \otimes \theta)(T_k). \tag{2.5}$$

First note that (2.5) follows from (2.3) when k is of the form $h\otimes a$ for some $h \in L^2(Y\times X)$ and $a \in A$; by linearity (2.5) holds for all $k \in L^2(Y\times X)\odot A$. Now, by Proposition 1.79, there is a sequence $(k_i)_{i\in\mathbb{N}} \subseteq L^2(Y\times X)\odot A$ such that $\|k_i - k\|_2 \to 0$. Thus we also have $\|\theta \circ k_i - \theta \circ k\|_2 \to 0$. By Lemma 2.1 we have $T_{k_i} \to T_k$ and $T_{\theta\circ k_i} \to T_{\theta\circ k}$ in operator norm. Since $T_{k_i} \to T_k$ we have $(\mathrm{id}\otimes\theta)(T_{k_i}) \to (\mathrm{id}\otimes\theta)(T_k)$. Now, since we have already shown that (2.5) holds for all $k \in L^2(Y \times X) \odot A$, we have $T_{\theta\circ k_i} = (\mathrm{id} \otimes \theta)(T_{k_i})$; thus (2.5) follows.

It follows from (2.5) and the definition of φ^θ that

$$(\mathrm{id} \otimes \theta)\big(S_\varphi(T_k)\big) = S_{\varphi^\theta}(T_{\theta\circ k}) = S_{\varphi^\theta}\big((\mathrm{id} \otimes \theta)(T_k)\big), \quad k \in L^2(Y \times X, A);$$

hence

$$S_{\varphi^\theta} \circ (\mathrm{id} \otimes \theta) = (\mathrm{id} \otimes \theta) \circ S_\varphi. \tag{2.6}$$

Thus S_φ is completely bounded if and only if S_{φ^θ} is, and in this case $\|\varphi\|_{\mathfrak{S}} = \|\varphi^\theta\|_{\mathfrak{S}}$. $\square$

Our aim is to characterise Schur A-multipliers analogously to Theorem 1.58. To begin we need some preliminary results.

Lemma 2.7. *Let $\varphi \in \mathfrak{S}(X, Y; A, \mathcal{B}(\mathcal{H}))$, $C \in \mathcal{D}_X$, $D \in \mathcal{D}_Y$, and suppose that $T \in \mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} A$. Then*

$$S_\varphi\big((D \otimes I_{\mathcal{H}})T(C \otimes I_{\mathcal{H}})\big) = (D \otimes I_{\mathcal{H}})S_\varphi(T)(C \otimes I_{\mathcal{H}}). \tag{2.7}$$

Proof. We show that (2.7) holds when $T = T_k \otimes a$ for some $a \in A$ and $k \in L^2(Y \times X)$; the result will then follow from linearity and continuity. Take $c \in L^\infty(X)$ and $d \in L^\infty(Y)$ such that $C = M_c$ and $D = M_d$. Take $f \in L^2(X)$, $\eta \in \mathcal{H}$ and $y \in Y$, then

$$\begin{aligned}(D \otimes I_{\mathcal{H}})(T_k \otimes a)(C \otimes I_{\mathcal{H}})(f \otimes \eta)(y) &= d(y)\Big(\int_X k(y,x)f(x)c(x)\,dx\Big)a\eta \\ &= (T_{k_{c,d}\otimes a})(f \otimes \eta)(y),\end{aligned}$$

where $k_{c,d}(y,x) = c(x)d(y)k(y,x)$ $(x \in X,\ y \in Y)$. Hence $S_\varphi((D \otimes I_{\mathcal{H}})(T_k \otimes a)(C \otimes I_{\mathcal{H}})) = T_{\varphi\cdot(k_{c,d}\otimes a)}$. On the other hand,

$$\begin{aligned}(D \otimes I_{\mathcal{H}})S_\varphi(T_k \otimes a)(C \otimes I_{\mathcal{H}})(f \otimes \eta)(y) &= d(y)\Big(\int_X k(y,x)c(x)f(x)\varphi(x,y)(a)\eta\,dx\Big) \\ &= T_{\varphi\cdot(k_{c,d}\otimes a)}(f \otimes \eta)(y).\end{aligned}$$

Equation (2.7) now follows from the density of sums of elementary tensors in $L^2(X) \otimes \mathcal{H}$ and the linearity and continuity of operators of the form T_k. This completes the proof. $\square$

Lemma 2.8. *Let $\mathcal{E}$ be a separable Hilbert space, and let $(\theta, \mathcal{H}_\theta)$ be a non-degenerate representation of $\mathcal{K}(\mathcal{E}) \otimes_{\min} A$. Then there exists a non-degenerate representation $(\rho, \mathcal{H}_\rho)$ of A on a separable Hilbert space, and a unitary operator $U : \mathcal{H}_\theta \to \mathcal{E} \otimes \mathcal{H}_\rho$, such that*

$$U\theta(b \otimes a)U^* = b \otimes \rho(a), \quad a \in A,\ b \in \mathcal{K}(\mathcal{E}).$$

Proof. Let $\mathrm{M}(\mathcal{K}(\mathcal{E}) \otimes_{\min} A)$ denote the multiplier algebra of $\mathcal{K}(\mathcal{E}) \otimes_{\min} A$. By Pedersen [44, Proposition 3.12.10] there exists a unital $*$-homomorphism $\hat{\theta} : \mathrm{M}(\mathcal{K}(\mathcal{E}) \otimes_{\min} A) \to \mathcal{B}(\mathcal{H}_\theta)$ extending θ. The map

$$\mathcal{K}(\mathcal{E}) \to \mathcal{B}(\mathcal{H}_\theta);\ b \mapsto \hat{\theta}(b \otimes I_{\mathcal{H}}), \quad b \in \mathcal{K}(\mathcal{E}),$$

is clearly a non-degenerate representation of $\mathcal{K}(\mathcal{E})$ on $\mathcal{H}_\theta$. By Arveson [3, page 20, Corollary 1] there exists a separable Hilbert space $\mathcal{H}_\rho$, and a unitary operator $U : \mathcal{H}_\theta \to \mathcal{E} \otimes \mathcal{H}_\rho$, such that

$$U\hat{\theta}(b \otimes I_{\mathcal{H}})U^* = b \otimes I_{\mathcal{H}_\rho}, \quad b \in \mathcal{K}(\mathcal{E}). \tag{2.8}$$

Define $\tilde{\theta} : \mathrm{M}(\mathcal{K}(\mathcal{E}) \otimes_{\min} A) \to \mathcal{B}(\mathcal{E} \otimes \mathcal{H}_\rho)$ by

$$\tilde{\theta}(T) := U\hat{\theta}(T)U^*, \quad T \in \mathrm{M}(\mathcal{K}(\mathcal{E}) \otimes_{\min} A).$$

For $a \in A$ and $b \in B$ we have

$$\begin{aligned}
\tilde{\theta}(b \otimes I_{\mathcal{H}})\tilde{\theta}(I_{\mathcal{E}} \otimes a) &= U\hat{\theta}(b \otimes I_{\mathcal{H}})U^*U\hat{\theta}(I_{\mathcal{E}} \otimes a)U^* \\
&= U\hat{\theta}(b \otimes a)U^* \\
&= U\hat{\theta}\big((I_{\mathcal{E}} \otimes a)(b \otimes I_{\mathcal{H}})\big)U^* \\
&= U\hat{\theta}(I_{\mathcal{E}} \otimes a)U^*U\hat{\theta}(b \otimes I_{\mathcal{H}})U^* \\
&= \tilde{\theta}(I_{\mathcal{E}} \otimes a)\tilde{\theta}(b \otimes I_{\mathcal{H}}).
\end{aligned}$$

Since $\tilde{\theta}(I_{\mathcal{E}} \otimes a)$ and $\tilde{\theta}(b \otimes I_{\mathcal{H}})$ commute it follows from (2.8), and Tomita's Commutation Theorem [8, Theorem III.4.5.8], that we must have $\tilde{\theta}(I_{\mathcal{E}} \otimes a) = I_{\mathcal{E}} \otimes \rho(a)$ for some $\rho(a) \in \mathcal{B}(\mathcal{H}_\rho)$. It follows that

$$\rho : A \to \mathcal{B}(\mathcal{H}_\rho);\ a \mapsto \rho(a),$$

is a non-degenerate representation of A on $\mathcal{H}_\rho$. Indeed, the $*$-homomorphism property for ρ is immediate from that of $\tilde{\theta}$. Moreover, if $b \in \mathcal{K}(\mathcal{E})$ and $a \in A$ then

$$\begin{aligned} U\theta(b \otimes a)U^* = U\hat{\theta}(b \otimes I_{\mathcal{H}})\hat{\theta}(I_{\mathcal{E}} \otimes a)U^* &= (b \otimes I_{\mathcal{H}_\rho})U\hat{\theta}(I_{\mathcal{E}} \otimes a)U^* \\ &= (b \otimes I_{\mathcal{H}_\rho})\tilde{\theta}(I_{\mathcal{E}} \otimes a) \\ &= (b \otimes I_{\mathcal{H}_\rho})\big(I_{\mathcal{E}} \otimes \rho(a)\big) \\ &= b \otimes \rho(a), \end{aligned}$$

as required. The non-degeneracy of ρ is now immediate from that of θ. □

Now we are able to characterise Schur A-multipliers.

Theorem 2.9. *Let $\varphi : X \times Y \to \mathcal{CB}(A, \mathcal{B}(\mathcal{H}))$ be a bounded, pointwise-measurable function. The following are equivalent:*

i. φ is a Schur A-multiplier;

ii. there exists a non-degenerate representation $(\rho, \mathcal{H}_\rho)$ of A on a separable Hilbert space, and operators $V \in L^\infty(X, \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho))$ and $W \in L^\infty(Y, \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho))$, such that

$$\varphi(x, y)(a) = W(y)^*\rho(a)V(x), \quad a \in A,$$

for almost all $(x, y) \in X \times Y$.

Moreover, when the conditions hold the functions V and W may be chosen so that

$$\|\varphi\|_{\mathfrak{S}} = \operatorname*{ess\,sup}_{x \in X} \|V(x)\| \operatorname*{ess\,sup}_{y \in Y} \|W(y)\|.$$

Proof. (i)⇒(ii) Suppose that φ is a Schur A-multiplier. Let $\mathcal{E} = L^2(X) \oplus L^2(Y)$ and define $\Phi : \mathcal{K}(\mathcal{E}) \otimes_{\min} A \to \mathcal{K}(\mathcal{E}) \otimes_{\min} \mathcal{B}(\mathcal{H})$ by

$$\Phi\left(\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \otimes a\right) = \begin{pmatrix} 0 & S_\varphi(x_{1,2} \otimes a) \\ 0 & 0 \end{pmatrix}, \quad (x_{i,j})_{i,j=1}^2 \in \mathcal{K}(\mathcal{E}),\ a \in A. \tag{2.9}$$

It is immediate that Φ is a completely bounded map with $\|\Phi\|_{\mathrm{cb}} = \|S_\varphi\|_{\mathrm{cb}}$. By the Haagerup–Paulsen–Wittstock Theorem, Theorem 1.18, there exists a Hilbert space $\mathcal{L}$, a non-degenerate representation $\theta : \mathcal{K}(\mathcal{E}) \otimes_{\min} A \to \mathcal{B}(\mathcal{L})$, and operators $V_0, W_0 \in \mathcal{B}(\mathcal{E} \otimes \mathcal{H}, \mathcal{L})$, such that

$$\Phi(T) = W_0^* \theta(T) V_0, \quad T \in \mathcal{K}(\mathcal{E}) \otimes_{\min} A.$$

It follows, by the construction used in the proof of Theorem 1.18, that $\mathcal{L}$ is separable since $\mathcal{K}(\mathcal{E}) \otimes_{\min} A$ is separable. By Lemma 2.8 there exist a unitary operator $U : \mathcal{L} \to \mathcal{E} \otimes \mathcal{H}$, and a representation $(\rho, \mathcal{H}_\rho)$ of A on a separable Hilbert space, such that

$$U\theta(b \otimes a)U^* = b \otimes \rho(a), \quad a \in A,\ b \in \mathcal{K}(\mathcal{E}).$$

Let $V_1 = UV_0$ and $W_1 = UW_0$, so that

$$\begin{aligned} \Phi(b \otimes a) &= W_0 \theta(b \otimes a) V_0 \\ &= W_0^* U^* \big(b \otimes \rho(a)\big) U V_0 \\ &= W_1^* \big(b \otimes \rho(a)\big) V_1, \end{aligned}$$

for all $b \in \mathcal{K}(\mathcal{E}),\ a \in A$. Recall that $\mathcal{E} = L^2(X) \oplus L^2(Y)$, so that $V_1, W_1 \in \mathcal{B}(\mathcal{E} \otimes \mathcal{H}, \mathcal{E} \otimes \mathcal{H}_\rho)$ can be written as 2×2 matrices acting on a column vector with the first entry in $L^2(X, \mathcal{H})$ and the second in $L^2(Y, \mathcal{H})$, with range a column vector with first entry in $L^2(X, \mathcal{H}_\rho)$, second entry in $L^2(Y, \mathcal{H}_\rho)$. Comparing this matrix form with (2.9) we obtain bounded operators $V_2 : L^2(X, \mathcal{H}) \to L^2(X, \mathcal{H}_\rho)$ and $W_2 : L^2(Y, \mathcal{H}) \to L^2(Y, \mathcal{H}_\rho)$ (namely, the lower left entry of V_1 and the upper right entry of W_1 respectively) such that

$$S_\varphi(b \otimes a) = W_2^* \big(b \otimes \rho(a)\big) V_2, \quad a \in A,\ b \in \mathcal{K}(L^2(X), L^2(Y)). \tag{2.10}$$

Let

$$\mathcal{S} := \overline{\operatorname{span}}\{TV_2 L^2(X, \mathcal{H}) : T \in \mathcal{K}(L^2(X)) \otimes_{\min} \rho(A)\}.$$

Clearly $\mathcal{S}$ is invariant under $\mathcal{K}(L^2(X)) \otimes_{\min} \rho(A)$, so the projection onto $\mathcal{S}$ commutes with $\mathcal{K}(L^2(X)) \otimes_{\min} \rho(A)$; by [8, Theorem III.4.5.8] this projection has the form $I_{L^2(X)} \otimes$

E for some projection $E \in \rho(A)'$. Moreover,

$$V_2 = (I_{L^2(X)} \otimes E)V_2. \tag{2.11}$$

Set

$$\tilde{\rho} := \mathrm{id} \otimes \rho : \mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} A \to \mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} \rho(A);$$

by (2.10) and (2.11) we have

$$S_\varphi(T) = W_2^* \tilde{\rho}(T)(I_{L^2(X)} \otimes E)V_2, \quad T \in \mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} A. \tag{2.12}$$

If $c \in L^\infty(X)$ and $d \in L^\infty(Y)$ then clearly

$$\tilde{\rho}\big((M_d^* \otimes I_{\mathcal{H}})T(M_c \otimes I_{\mathcal{H}})\big) = (M_d^* \otimes I_{\mathcal{H}_\rho})\tilde{\rho}(T)(M_c \otimes I_{\mathcal{H}_\rho}). \tag{2.13}$$

Let $W = (I_{L^2(Y)} \otimes E)W_2$. Since

$$\tilde{\rho}(T)(I_{L^2(Y)} \otimes E) = (I_{L^2(Y)} \otimes E)\tilde{\rho}(T)$$

we conclude from (2.12) that

$$S_\varphi(T) = W_2^*(I_{L^2(Y)} \otimes E)\tilde{\rho}(T)V_2 = W^* \tilde{\rho}(T)V_2, \quad T \in \mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} A. \tag{2.14}$$

Lemma 2.7, and identities (2.13) and (2.14), imply that

$$W^*(M_d^* \otimes I_{\mathcal{H}_\rho})\tilde{\rho}(T)V_2 = (M_d^* \otimes I_{\mathcal{H}})W^* \tilde{\rho}(T)V_2, \quad T \in \mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} A. \tag{2.15}$$

Thus

$$\big\langle \tilde{\rho}(T)V_2\xi, (M_d \otimes I_{\mathcal{H}_\rho})W\eta \big\rangle = \langle \tilde{\rho}(T)V_2\xi, W(M_d \otimes I_{\mathcal{H}})\eta \rangle,$$

for all $\xi \in L^2(X, \mathcal{H}),\ \eta \in L^2(Y, \mathcal{H})$. We conclude, using the definition of $\mathcal{S}$, that

$$(I_{L^2(Y)} \otimes E)(M_d \otimes I_{\mathcal{H}_\rho})W = (I_{L^2(Y)} \otimes E)W(M_d \otimes I_{\mathcal{H}}),$$

hence $(M_d \otimes I_{\mathcal{H}_\rho})W = W(M_d \otimes I_{\mathcal{H}})$ for all $d \in L^\infty(Y)$. It follows from Theorem 1.82 and Remark 1.83 that $W \in L^\infty(Y, \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho))$. Now let

$$\mathcal{T} := \overline{\operatorname{span}}\{TWL^2(Y, \mathcal{H}) : T \in \mathcal{K}(L^2(Y)) \otimes \rho(A)\}.$$

As with $\mathcal{S}$ above, the projection onto $\mathcal{T}$ has the form $I_{L^2(Y)} \otimes F$ for some projection $F \in \rho(A)'$. Setting $V = (I_{L^2(X)} \otimes F)V_2$ and arguing similarly to above one obtains $(M_c \otimes I_{\mathcal{H}_\rho})V = V(M_c \otimes I_{\mathcal{H}})$ for all $c \in L^\infty(X)$, and hence that $V \in L^\infty(X, \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho))$. Note that $W = (I_{L^2(Y)} \otimes F)W$ so, by (2.14),

$$S_\varphi(T) = W^*(I_{L^2(Y)} \otimes F)\tilde{\rho}(T)V_2 = W^*\tilde{\rho}(T)(I_{L^2(Y)} \otimes F)V_2 = W^*\tilde{\rho}(T)V, \tag{2.16}$$

for every $T \in \mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} A$.

Let $k \in L^2(Y \times X)$ and $a \in A$; for $\xi \in L^2(X, \mathcal{H})$ and $\eta \in L^2(Y, \mathcal{H})$ we have

$$\langle S_\varphi(T_k \otimes a)\xi, \eta\rangle = \int_Y \int_X k(y,x) \left\langle \varphi(x,y)(a)\big(\xi(x)\big), \eta(y)\right\rangle \, d\mu(x)\, d\nu(y). \tag{2.17}$$

Using (2.16) we also have

$$\begin{aligned}\langle S_\varphi(T_k \otimes a)\xi, \eta\rangle &= \left\langle W^*\big(T_k \otimes \rho(a)\big)V\xi, \eta\right\rangle \\ &= \left\langle \big(T_k \otimes \rho(a)\big)V\xi, W\eta\right\rangle \\ &= \int_Y \int_X k(y,x) \left\langle \rho(a)\big(V(x)\xi(x)\big), W(y)\eta(y)\right\rangle \, d\mu(x)\, d\nu(y) \\ &= \int_Y \int_X k(y,x) \left\langle W(y)^*\rho(a)V(x)\xi(x), \eta(y)\right\rangle \, d\mu(x)\, d\nu(y)\end{aligned} \tag{2.18}$$

Comparing this with (2.17), and using the fact that both identities hold for all $k \in L^2(Y \times X)$, we conclude

$$\left\langle \varphi(x,y)(a)\big(\xi(x)\big), \eta(y)\right\rangle = \langle W(y)^*\rho(a)V(x)\xi(x), \eta(y)\rangle \tag{2.19}$$

almost everywhere, for all $\xi \in L^2(X, \mathcal{H})$ and $\eta \in L^2(Y, \mathcal{H})$. If the measures μ and ν are finite we have, in particular, that the above applies when $\xi = \chi_X \otimes e_i \in L^2(X, \mathcal{H})$ and $\eta = \chi_Y \otimes e_j \in L^2(Y, \mathcal{H})$ for any elements e_i, e_j of a countable orthonormal basis for $\mathcal{H}$.

It follows that

$$\varphi(x,y)(a) = W(y)^*\rho(a)V(x) \text{ for almost all } (x,y) \in X \times Y,$$

as claimed. If the measures μ and ν are not finite use the σ-finiteness of (X,μ) and (Y,ν) to choose increasing sequences $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ of subsets of X and Y respectively, with each of the terms having finite measure. Take $(x,y) \in X \times Y$, find $n \in \mathbb{N}$ such that $(x,y) \in X_n \times Y_n$, and let $\xi_{i,n} = \chi_{X_n} \otimes e_i$ and $\eta_{j,n} = \chi_{Y_n} \otimes e_j$. Now (2.19) holds with $\xi = \xi_{i,n}$ and $\eta = \eta_{j,n}$ for all $i,j \in \mathbb{N}$, so we conclude as above using the fact that X_n and Y_n have finite measure.

(ii) $\Rightarrow$ (i) By (2.17) and (2.18) condition (ii) implies that the map $S_\varphi : \mathcal{S}_2(X,Y;A) \to \mathcal{S}_2(X,Y;\mathcal{B}(\mathcal{H}))$ satisfies

$$S_\varphi(T_h \otimes a) = W^*\big(T_h \otimes \rho(a)\big)V, \quad h \in L^2(Y \times X),\ a \in A.$$

By linearity

$$S_\varphi(T_k) = W^* T_{\rho\circ k} V, \quad k \in L^2(Y \times X) \odot A. \tag{2.20}$$

Now let $k \in L^2(Y \times X, A)$ be arbitrary. By Proposition 1.79, and the remarks in Subsection 1.5.2, there is a sequence $(k_i)_{i\in\mathbb{N}} \subseteq L^2(Y \times X) \odot A$ with $\|k_i - k\|_2 \to 0$. Using (2.5), (2.20), Lemma 2.1, and the fact that φ is bounded, we obtain

$$\begin{aligned} S_\varphi(T_k) = \lim_{i\to\infty} S_\varphi(T_{k_i}) &= \lim_{i\to\infty} W^* T_{\rho\circ k_i} V \\ &= W^*\big(\lim_{i\to\infty} T_{\rho\circ k_i}\big)V \\ &= W^*\big(\lim_{i\to\infty} \tilde{\rho}(T_{k_i})\big)V \\ &= W^*\tilde{\rho}(T_k)V. \end{aligned}$$

Thus the map $T \mapsto W^*\tilde{\rho}(T)V$ is an extension of S_φ to a completely bounded map on $\mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} A$, so φ is a Schur A-multiplier.

Finally, for the norm equality, note that from Theorem 1.18 the operators V_0 and W_0 appearing in the above proof can be chosen so that $\|S_\varphi\|_{\mathrm{cb}} = \|V_0\|\|W_0\|$, and that the operators V and W have the same norm as V_0 and W_0 respectively, from which they were obtained. We have, by equation (6) of [55, page 259],

$$\operatorname*{ess\ sup}_{x\in X} \|V(x)\| = \|V\| = \|V_0\|,$$

and similarly $\text{ess sup}_{y\in Y} \|W(y)\| = \|W_0\|$; it follows that

$$\operatorname*{ess\ sup}_{x\in X} \|V(x)\| \operatorname*{ess\ sup}_{y\in Y} \|W(y)\| = \|V_0\|\|W_0\| = \|S_\varphi\|_{\mathrm{cb}} = \|\varphi\|_{\mathfrak{S}}. \qquad \square$$

Remarks 2.10. (i) In the case $A = \mathbb{C}$ Theorem 2.9 reduces to the characterisation of Schur multipliers in Theorem 1.58. Indeed, if $A = \mathbb{C}$ then, by Remark 2.4, condition (i) of Theorem 2.9 simply says that φ is a Schur multiplier in the sense of Definition 1.56. Since every C^*-algebra representation of $\mathbb{C}$ is an inflation of the identity representation condition (ii), when $A = \mathbb{C}$, says that there exist a representation $\rho : \mathbb{C} \to \mathcal{B}(\ell^2)$, and operators $V \in L^\infty(X, \mathcal{B}(\mathbb{C}, \ell^2))$ and $W \in L^\infty(Y, \mathcal{B}(\mathbb{C}, \ell^2))$, such that $\varphi(x, y)(a) = W(y)^*\rho(a)V(x)$ ($a \in \mathbb{C}$, almost all $(x, y) \in X \times Y$). Identifying $\mathcal{B}(\mathbb{C}, \ell^2)$ with ℓ^2 we have $V : X \to \ell^2$ and $W : Y \to \ell^2$, so that

$$\varphi(x, y) = \langle V(x), W(y)\rangle,$$

for almost all $(x, y) \in X \times Y$.

(ii) Suppose that φ is a Schur A-multiplier and take the operators V and W, and the representation $(\rho, \mathcal{H}_\rho)$ of A, provided by Theorem 2.9. By Effros–Ruan [22, Proposition 8.1.5] the map $T \mapsto W^*(\mathrm{id} \otimes \rho)(T)V$ extends to a completely bounded map

$$\Psi : \mathcal{B}(L^2(X), L^2(Y)) \otimes_{\min} A \to \mathcal{B}(L^2(X), L^2(Y)) \otimes_{\min} \mathcal{B}(\mathcal{H}),$$

which clearly extends S_φ. So if $\varphi : X \times Y \to \mathcal{CB}(A, \mathcal{B}(\mathcal{H}))$ is a Schur A-multiplier then the map S_φ automatically extends to a completely bounded map defined on $\mathcal{B}(L^2(X), L^2(Y)) \otimes_{\min} A$.

Recall from Subsection 1.1.3 that if $\mathcal{M}$ is a Hilbert C^*-bimodule then we denote the C^*-algebra valued inner product on $\mathcal{M}$ by $\langle\cdot|\cdot\rangle$.

Theorem 2.11. *Suppose that $\varphi : X \times Y \to \mathcal{CB}(A)$ is a bounded, pointwise-measurable function. Consider the conditions:*

i. there exists a countably-generated Hilbert A-bimodule $\mathcal{M}$ and essentially bounded, measurable functions $v : X \to \mathcal{M}$ and $w : Y \to \mathcal{M}$ such that

$$\varphi(x, y)(a) = \langle w(y)|a \cdot v(x)\rangle , \quad a \in A,$$

for almost all $(x, y) \in X \times Y$;

ii. φ is a Schur A-multiplier.

Then (i) implies (ii). If A is finite-dimensional then (i) and (ii) are equivalent.[1]

Proof. Suppose condition (i) holds. It follows from [21, Example 2.8] that there exist a faithful representation $(\phi, \mathcal{H}_\phi)$ of A, and an isometry $\tau : \mathcal{M} \to \mathcal{B}(\mathcal{H}, \mathcal{H}_\phi)$, such that

$$\tau(a \cdot z) = \phi(a)\tau(z),$$
$$\phi\big(\langle y|z\rangle \big) = \tau(y)^*\tau(z),$$

for all $a \in A,\ y, z \in \mathcal{M}$. Moreover, by construction, the Hilbert space $\mathcal{H}_\phi$ has dense subset $A \odot \mathcal{H}$, where $\mathcal{H}$ is the Hilbert space on which A is faithfully represented; we may therefore assume $\mathcal{H}_\phi$ is separable. We have that

$$\phi\big(\varphi(x, y)(a)\big) = \phi\big(\langle w(y)|a \cdot v(x)\rangle \big) = \tau\big(w(y)\big)^*\phi(a)\tau\big(v(x)\big),$$

for all $a \in A$, and almost all $(x, y) \in X \times Y$. Moreover, since v and w are measurable and τ is an isometry, the maps $\tau \circ v$ and $\tau \circ w$ are measurable. Since v and w are essentially bounded it follows that $\tau \circ v \in L^\infty(X, \mathcal{B}(\mathcal{H}, \mathcal{H}_\phi))$ and $\tau \circ w \in L^\infty(Y, \mathcal{B}(\mathcal{H}, \mathcal{H}_\phi))$. By Theorem 2.9 the map

$$\varphi^\phi : X \times Y \to \mathcal{CB}(\phi(A));\ \varphi^\phi(x, y)\big(\phi(a)\big) := \phi\big(\varphi(x, y)(a)\big), \quad a \in A,\ (x, y) \in X \times Y,$$

[1] This result was originally formulated as an equivalence for injective C^*-algebras; however, when injectivity is combined with our standing separability assumptions it turns out that we are considering only finite-dimensional C^*-algebras.

is a Schur $\phi(A)$-multiplier. By Proposition 2.6, φ is a Schur A-multiplier.

Now suppose that A is finite-dimensional. By Proposition 2.6 we may identify A with the C^*-algebra $\oplus_{k=1}^m M_{n_k}$, for some $m \in \mathbb{N}$, acting on the Hilbert space $\mathcal{H} = \oplus_{k=1}^m \mathbb{C}^{n_k}$. Suppose that φ is a Schur A-multiplier, so that by Theorem 2.9 there exists a representation $(\rho, \mathcal{H}_\rho)$ of A on a separable Hilbert space, and functions $V \in L^\infty(X, \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho))$ and $W \in L^\infty(Y, \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho))$, such that

$$\varphi(x, y)(a) = W(y)^* \rho(a) V(x), \quad a \in A,$$

for almost all $(x, y) \in X \times Y$. The space $\mathcal{B}(\mathcal{H}, \mathcal{H}_\rho)$ is an A-bimodule with respect to the actions

$$\begin{aligned} a \cdot T &= \rho(a) T, \\ T \cdot a &= Ta, \end{aligned}$$

for $a \in A,\ T \in \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho)$. Let $P_k \in \mathcal{B}(\mathcal{H})$ be the projection onto the direct summand $\mathbb{C}^{n_k}$ and define

$$\Psi(T) := \sum_{k=1}^m P_k T P_k, \quad T \in \mathcal{B}(\mathcal{H}),$$

which is a completely positive projection from $\mathcal{B}(\mathcal{H})$ onto A. As in Example 1.25 equip $\mathcal{B}(\mathcal{H}, \mathcal{H}_\rho)$ with the A-valued inner product

$$\langle S | T \rangle := \Psi(S^* T), \quad S, T \in \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho).$$

Since the projections P_k, $k \in \{1, \ldots, m\}$, are pairwise orthogonal and $\sum_{k=1}^m P_k = I_{\mathcal{H}}$ we have $\langle S | S \rangle = 0$ if and only if $S = 0$. We have

$$\langle S | T \cdot a \rangle = \Psi(S^* T a) = \Psi(S^* T) a = \langle S | T \rangle\, a, \quad a \in A,\ S, T \in \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho),$$

so that $\mathcal{M} := \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho)$ is a right inner product A-module. Moreover,

$$\langle a \cdot S | T \rangle = \Psi(S^* \rho(a)^* T) = \langle S | a^* \cdot T \rangle, \quad a \in A,\ S, T \in \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho).$$

Thus, the map $\phi_a : S \mapsto a \cdot S$ is adjointable and $a \mapsto \phi_a$ is a representation of A on $\mathcal{M}$. Thus the completion of $\mathcal{M}$ in the norm arising from $\langle \cdot | \cdot \rangle$ is a Hilbert A-bimodule and $\varphi(x,y)(a) = \langle W(y) | a \cdot V(x) \rangle$ for all $a \in A$ and almost all $(x,y) \in X \times Y$. As $\mathcal{H}$ is finite-dimensional and $\mathcal{H}_\rho$ is separable, $\mathcal{B}(\mathcal{H}, \mathcal{H}_\rho)$ has a countable dense set; it follows that $\mathcal{M}$ is a countably-generated Hilbert A-bimodule. □

If $\varphi \in L^\infty(X \times Y)$ is a classical Schur multiplier then the associated map S_φ is a weak* continuous, completely bounded map on $\mathcal{B}(L^2(X), L^2(Y))$. Due to the way Schur A-multipliers are defined, and the presence of the C^*-algebra A, Schur A-multipliers do not seem to possess a canonical weak* extension of this type. If φ is a Schur A-multiplier then, by definition, S_φ maps $\mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} A$ to $\mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} \mathcal{B}(\mathcal{H})$. One might hope that by taking the second dual of this map we would obtain something similar to the classical case; here we show one condition under which this idea is successful.

Proposition 2.12. *Suppose that A is a von Neumann algebra. If $\varphi \in \mathfrak{S}(X, Y; A, \mathcal{B}(\mathcal{H}))$ then the map S_φ has a unique extension to a completely bounded, weak*-continuous map*

$$\Psi_\varphi : \mathcal{B}(L^2(X), L^2(Y)) \mathbin{\overline{\otimes}} A^{**} \to \mathcal{B}(L^2(X), L^2(Y)) \mathbin{\overline{\otimes}} \mathcal{B}(\mathcal{H}).$$

Proof. Let $P : \mathcal{B}(\mathcal{H})^{**} \to \mathcal{B}(\mathcal{H})$ be the canonical projection; then the map $\mathrm{id} \otimes P : \mathcal{B}(L^2(X), L^2(Y)) \mathbin{\overline{\otimes}} \mathcal{B}(\mathcal{H})^{**} \to \mathcal{B}(L^2(X), L^2(Y)) \mathbin{\overline{\otimes}} \mathcal{B}(\mathcal{H})$ is weak*-continuous and completely contractive [55, Proposition IV.5.13].

Let P_X and P_Y denote the the projections from $L^2(X) \oplus L^2(Y)$ onto $L^2(X)$ and $L^2(Y)$ respectively. Define $\Phi : \mathcal{K}(L^2(X) \oplus L^2(Y)) \otimes_{\min} A \to \mathcal{K}(L^2(X) \oplus L^2(Y)) \otimes_{\min} \mathcal{B}(\mathcal{H})$ by

$$\Phi\left(\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \otimes a \right) = \begin{pmatrix} 0 & S_\varphi(x_{1,2} \otimes a) \\ 0 & 0 \end{pmatrix}.$$

By Huruya [32, Example 1], for any von Neumann algebra B, there is a canonical normal isomorphism

$$(\mathcal{K}(L^2(X) \oplus L^2(Y)) \otimes_{\min} B)^{**} \cong \mathcal{B}(L^2(X) \oplus L^2(Y)) \mathbin{\overline{\otimes}} B^{**}. \tag{2.21}$$

We may therefore view Φ^{**} as a completely bounded map from $\mathcal{B}(L^2(X) \oplus L^2(Y)) \overline{\otimes} A^{**}$ to $\mathcal{B}(L^2(X) \oplus L^2(Y)) \overline{\otimes} \mathcal{B}(\mathcal{H})^{**}$ extending Φ. As $(\mathcal{K}(L^2(X) \oplus L^2(Y)) \otimes_{\min} A$ is weak* dense in $\mathcal{B}(L^2(X) \oplus L^2(Y)) \overline{\otimes} A^{**}$ [18, Section 12.1] we have that for any $T \in \mathcal{B}(L^2(X) \oplus L^2(Y)) \overline{\otimes} A^{**}$ there is $\Psi(T) \in \mathcal{B}(L^2(X), L^2(Y)) \overline{\otimes} \mathcal{B}(\mathcal{H})^{**}$ such that

$$\Phi^{**}(T) = \begin{pmatrix} 0 & \Psi(T) \\ 0 & 0 \end{pmatrix}.$$

In particular,

$$\Phi^{**}\big((P_Y \otimes \mathrm{id})T(P_X \otimes \mathrm{id})\big) = \begin{pmatrix} 0 & \Psi\big((P_Y \otimes \mathrm{id})T(P_X \otimes \mathrm{id})\big) \\ 0 & 0 \end{pmatrix},$$

and thus the mapping

$$\tilde{\Psi} := \Psi\big|_{\mathcal{B}(L^2(X), L^2(Y)) \overline{\otimes} A^{**}} : \mathcal{B}(L^2(X), L^2(Y)) \overline{\otimes} A^{**} \to \mathcal{B}(L^2(X), L^2(Y)) \overline{\otimes} \mathcal{B}(\mathcal{H})^{**}$$

is completely bounded and weak*-continuous. Hence the composition

$$(\mathrm{id} \otimes P) \circ \tilde{\Psi} : \mathcal{B}(L^2(X), L^2(Y)) \overline{\otimes} A^{**} \to \mathcal{B}(L^2(X), L^2(Y)) \overline{\otimes} \mathcal{B}(\mathcal{H})$$

is a completely bounded, weak*-continuous map extending S_φ. It is a unique extension by weak* density of $\mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} A$ in $\mathcal{B}(L^2(X), L^2(Y)) \overline{\otimes} A^{**}$. $\square$

In general the question of when such extensions exist, in a way which respects the representation of A, seems rather difficult; we suggest the following definition for further study.

Definition 2.13. Let $(\theta, \mathcal{H}_\theta)$ be a faithful representation of A on a separable Hilbert space. A function $\varphi \in \mathfrak{S}_0(X, Y; A)$ will be called a *Schur θ-multiplier of A* if the map

$$S_{\varphi^\theta} : \mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} \theta(A) \to \mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} \theta(A)$$

extends to a weak*-continuous, completely bounded map on $\mathcal{B}(L^2(X), L^2(Y)) \overline{\otimes} \theta(A)''$.

Schur θ-multipliers will appear later in Corollary 3.14 and Proposition 4.4.

Chapter 3

Herz–Schur multipliers and transference

In this chapter we introduce Herz–Schur multipliers of a C^*-dynamical system, characterise them in the spirit of De Cannière–Haagerup, Theorem 1.61, and investigate the connection to the Schur A-multipliers studied above.

Throughout this section A denotes a separable C^*-algebra, which will be considered as a C^*-subalgebra of $\mathcal{B}(\mathcal{H}_A)$, where $\mathcal{H}_A$ is the Hilbert space of the universal representation of A. We also work throughout with a locally compact group G with a fixed left Haar measure, using the notation of Section 1.3. Finally, α denotes an action of G on A, *i.e.* $\alpha : G \to \mathrm{Aut}(A)$ is a group homomorphism which is continuous in the point-norm topology. In short, we work with a C^*-dynamical system (A, G, α) as described in Section 1.6.

3.1 Herz–Schur multipliers

Recall from Definition 1.76 that a bounded function $F : G \to \mathcal{B}(A)$ is called pointwise-measurable if, for each $a \in A$, the map $s \mapsto F(s)(a)$ is a measurable function from G to A. Let $F : G \to \mathcal{CB}(A)$ be a bounded, pointwise-measurable function and $f \in L^1(G, A)$.

Define

$$F \cdot f : G \to A;\ (F \cdot f)(s) := F(s)\big(f(s)\big), \quad s \in G.$$

We show that $F \cdot f \in L^1(G, A)$. That $F \cdot f$ is measurable follows from Lemma 1.77. Let $\|F\|_\infty := \sup_{s \in G} \|F(s)\|$; then

$$\begin{aligned}\int_G \|F \cdot f(s)\|\, ds &= \int_G \|F(s)\big(f(s)\big)\|\, ds \\ &\leq \|F\|_\infty \int_G \|f(s)\|\, ds \\ &= \|F\|_\infty \|f\|_1,\end{aligned}$$

so $F \cdot f \in L^1(G, A)$ and $\|F \cdot f\|_1 \leq \|F\|_\infty \|f\|_1$.

Definition 3.1. Let $F : G \to \mathcal{CB}(A)$ be a bounded, pointwise-measurable function. We say that F is a *Herz–Schur multiplier of the system* (A, G, α), or simply a *Herz–Schur* (A, G, α)*-multiplier*, if the map

$$S_F : (\pi \rtimes \lambda)\big(L^1(G, A)\big) \to (\pi \rtimes \lambda)\big(L^1(G, A)\big)$$

given by

$$S_F\big((\pi \rtimes \lambda)(f)\big) := \big((\pi \rtimes \lambda)(F \cdot f)\big), \quad f \in L^1(G, A), \tag{3.1}$$

is completely bounded. The space of all Herz–Schur (A, G, α)-multipliers will be denoted $\mathfrak{S}(A, G, \alpha)$, and endowed with the norm $\|F\|_{\mathrm{HS}} := \|S_F\|_{\mathrm{cb}}$.

If S_F is bounded, but not necessarily completely bounded, we say F is a *multiplier of* (A, G, α) and write $\|F\|_{\mathrm{m}} := \|S_F\|$.

If $F \in \mathfrak{S}(A, G, \alpha)$ then the map S_F extends to a completely bounded map on $A \rtimes_{\alpha,r} G$, which will again be denoted by S_F.

There is a natural algebra structure on $\mathfrak{S}(A, G, \alpha)$. Let $F_1, F_2 \in \mathfrak{S}(A, G, \alpha)$ and define

$$F_1 + F_2 : G \to \mathcal{CB}(A);\ (F_1 + F_2)(s) := F_1(s) + F_2(s), \quad s \in G,$$

and

$$F_1 F_2 : G \to \mathcal{CB}(A);\ (F_1 F_2)(s) := F_1(s) \circ F_2(s), \quad s \in G.$$

It is clear that $S_{F_1+F_2} = S_{F_1} + S_{F_2}$ and $S_{F_1 F_2} = S_{F_1} S_{F_2}$, so that $\mathfrak{S}(A, G, \alpha)$ is an algebra with respect to these operations.

Let us check that $\|\cdot\|_{\mathrm{HS}}$ does indeed define a norm. Since the assignment $F \mapsto S_F$ is linear and multiplicative we only need to check definiteness. It is clear that if $F : G \to \mathcal{CB}(A)$ is equal almost everywhere to the zero function then $S_F = 0$, so $\|F\|_{\mathrm{HS}} = 0$. On the other hand, if $\|F\|_{\mathrm{HS}} = 0$ then S_F is the zero map; since $\pi \rtimes \lambda$ is a faithful representation of $L^1(G, A)$ this implies $F \cdot f \in L^1(G, A)$ is equal almost everywhere to the zero function. Since this holds for every $f \in L^1(G, A)$ we must have that F is zero almost everywhere.

Remarks 3.2. (i) Let H be a locally compact group and consider the C^*-dynamical system $(\mathbb{C}, H, \mathrm{id})$, where id denotes the identity action of H on $\mathbb{C}$. By Williams [60, Example 7.9] we have $\mathbb{C} \rtimes_{\mathrm{id},r} H = C^*_\lambda(H)$. Take $F \in \mathfrak{S}(\mathbb{C}, H, \mathrm{id})$ and consider F as a map from H to $\mathbb{C}$ under the natural identification of $\mathcal{CB}(\mathbb{C})$ with $\mathbb{C}$. Since $F \in \mathfrak{S}(\mathbb{C}, H, \mathrm{id})$ we have by definition that

$$S_F : \lambda^H\big(L^1(H)\big) \to \lambda^H\big(L^1(H)\big);\ S_F\big(\lambda^H(f)\big) = \lambda^H(F \cdot f) = \lambda^H(Ff), \quad f \in L^1(H),$$

is completely bounded, *i.e.* S_F satisfies equation (1.9). By Theorem 1.61, the existence of such a completely bounded map on $C^*_\lambda(H)$ is equivalent to F being a classical Herz–Schur multiplier; we conclude that classical Herz–Schur multipliers on H coincide with Herz–Schur $(\mathbb{C}, H, \mathrm{id})$-multipliers.

(ii) Recall from Theorem 1.93 that if $(\theta, \mathcal{H}_\theta)$ is a faithful representation of A then $A \rtimes_{\alpha,r} G$ is isomorphic to the closure of $(\pi^\theta \rtimes \lambda^\theta)(L^1(G, A))$. It follows that a bounded, pointwise-measurable function $F : G \to \mathcal{CB}(A)$ is a Herz–Schur (A, G, α)-multiplier if and only if the map

$$S_F^\theta : (\pi^\theta \rtimes \lambda^\theta)\big(L^1(G, A)\big) \to (\pi^\theta \rtimes \lambda^\theta)\big(L^1(G, A)\big)$$

given by

$$S_F^\theta\big((\pi^\theta \rtimes \lambda^\theta)(f)\big) := \big((\pi^\theta \rtimes \lambda^\theta)(F \cdot f)\big), \quad f \in L^1(G, A),$$

is completely bounded. Therefore we may define Herz–Schur (A, G, α)-multipliers through any faithful representation of A.

We now investigate some technicalities in the theory of Herz–Schur multipliers of a C^*-dynamical system concerning the weak* topology. As was discussed in (1.9), the map S_u on $C^*_\lambda(G)$ associated to a Herz–Schur multiplier u automatically has an extension to a completely bounded, weak*-continuous map on the weak* closure. Such extensions may not exist for Herz–Schur (A, G, α)-multipliers, so we make the following definition. Recall that $A \rtimes^{\mathrm{w}^*}_{\alpha,\theta} G$ denotes the weak*-closure of $A \rtimes_{\alpha,\theta} G$.

Definition 3.3. Let (A, G, α) be a C^*-dynamical system and $(\theta, \mathcal{H}_\theta)$ a faithful representation of A. A function $F : G \to \mathcal{CB}(A)$ will be called a *θ-multiplier* of (A, G, α) if the map

$$\Phi^\theta_F : \pi^\theta(a)\lambda^\theta_t \mapsto \pi^\theta\big(F(t)(a)\big)\lambda^\theta_t, \quad t \in G,\ a \in A,$$

extends to a bounded, weak*-continuous, function on $A \rtimes^{\mathrm{w}^*}_{\alpha,\theta} G$. Let $\|F\|_{\mathrm{w}} := \|\Phi^\theta_F\|$.

A θ-multiplier F will be called a *Herz–Schur θ-multiplier of* (A, G, α) if the extension of Φ^θ_F to $A \rtimes^{\mathrm{w}^*}_{\alpha,\theta} G$ is completely bounded. Let $\|F\|_{\mathrm{HSw}} := \|\Phi^\theta_F\|_{\mathrm{cb}}$.

When the extensions in Definition 3.3 exist they will be denoted by the same symbol Φ^θ_F. Observe that pointwise-measurability is not required in Definition 3.3; the following shows that θ-multipliers act similarly to the multipliers defined in Definition 3.1, at least when paired with a weak*-continuous functional.

Remark 3.4. Let (A, G, α) be a C^*-dynamical system and let $(\theta, \mathcal{H}_\theta)$ be a faithful representation of A. Recall that A is assumed to be a separable C^*-algebra. Suppose that $F : G \to \mathcal{B}(A)$ is a bounded map and $\Phi : A \rtimes^{\mathrm{w}^*}_{\alpha,\theta} G \to A \rtimes^{\mathrm{w}^*}_{\alpha,\theta} G$ is a bounded, weak*-continuous, map such that, for almost all $t \in G$,

$$\Phi\big(\pi^\theta(a)\lambda^\theta_t\big) = \pi^\theta\big(F(t)(a)\big)\lambda^\theta_t, \quad a \in A.$$

Then, for any $\omega \in \mathcal{B}(L^2(G, \mathcal{H}_\theta))_*$ and any $f \in L^1(G, A)$, the scalar-valued function on G given by $s \mapsto \big\langle \pi^\theta(F(s)(f(s)))\lambda^\theta_s, \omega \big\rangle$ is measurable, and

$$\Big\langle \Phi\big((\pi^\theta \rtimes \lambda^\theta)(f)\big), \omega \Big\rangle = \int_G \Big\langle \pi^\theta\Big(F(s)\big(f(s)\big)\Big)\lambda^\theta_s, \omega \Big\rangle \, ds, \quad f \in L^1(G, A).$$

Proof. Let $\omega \in \mathcal{B}(L^2(G, \mathcal{H}_\theta)_*$ and $f \in L^1(G, A)$. Since $s \mapsto \langle \pi^\theta(f(s))\lambda^\theta_s, \Phi_*(\omega)\rangle$ is measurable so is $s \mapsto \langle \pi^\theta(F(s)(f(s)))\lambda^\theta_s, \omega\rangle$. Now

$$\begin{aligned}\Big\langle \Phi\Big(\int_G \pi^\theta\big(f(s)\big)\lambda^\theta_s\,ds\Big), \omega\Big\rangle &= \Big\langle \int_G \pi^\theta\big(f(s)\big)\lambda^\theta_s\,ds, \Phi_*(\omega)\Big\rangle \\ &= \int_G \Big\langle \pi^\theta\big(f(s)\big)\lambda^\theta_s, \Phi_*(\omega)\Big\rangle\,ds \\ &= \int_G \Big\langle \Phi\Big(\pi^\theta\big(f(s)\big)\lambda^\theta_s\Big), \omega\Big\rangle\,ds \\ &= \int_G \Big\langle \pi^\theta\Big(F(s)\big(f(s)\big)\Big)\lambda^\theta_s, \omega\Big\rangle\,ds,\end{aligned}$$

as required. □

For the rest of this chapter we assume the topology on G is second-countable; this means that when equipped with Haar measure G is a standard measure space — see the discussion before Remark 1.42.

A success of the introduction of θ-multipliers is that we can recover a version of the result of De Cannière–Haagerup, Theorem 1.62. Let Γ be another locally compact group and $\phi : G \to X$ a (possibly vector-valued) function on G. We define

$$\phi^\Gamma : \Gamma \times G \to X;\ \phi^\Gamma(\gamma, t) := \phi(t), \quad \gamma \in \Gamma,\ t \in G.$$

Lemma 3.5. *Let (A, G, α) be a C^*-dynamical system, $(\theta, \mathcal{H}_\theta)$ a faithful representation of A, and Γ a locally compact group. Then*

$$A \rtimes^{\mathrm{w}^*}_{\alpha^\Gamma,\theta} (\Gamma \times G) = \mathrm{vN}(\Gamma) \mathbin{\overline{\otimes}} (A \rtimes^{\mathrm{w}^*}_{\alpha,\theta} G).$$

Proof. Define a representation of A on $L^2(\Gamma \times G, \mathcal{H}_\theta)$ by

$$\pi^\theta_\Gamma : A \to \mathcal{B}(L^2(\Gamma \times G, \mathcal{H}_\theta));\ \pi^\theta_\Gamma(a)\xi(\gamma, t) := \alpha^\Gamma_{(\gamma^{-1}, t^{-1})}(a)\big(\xi(\gamma, t)\big),$$

for all $a \in A,\ \xi \in L^2(\Gamma \times G, \mathcal{H}_\theta),\ \gamma \in \Gamma,\ t \in G$. Under the usual identification of $L^2(\Gamma \times G, \mathcal{H}_\theta)$ with $L^2(\Gamma) \otimes L^2(G, \mathcal{H}_\theta)$ we obtain

$$\pi^\theta_\Gamma(a) = I_{L^2(\Gamma)} \otimes \pi^\theta(a), \quad a \in A. \tag{3.2}$$

Define a unitary representation of $\Gamma \times G$ on $L^2(\Gamma) \otimes L^2(G, \mathcal{H}_\theta)$ by

$$\lambda_{(\gamma,t)} := \lambda_\gamma^\Gamma \otimes \lambda_t^\theta, \quad \gamma \in \Gamma,\ t \in G. \tag{3.3}$$

Since $(\pi^\theta, \lambda^\theta)$ is a covariant pair it is immediate that $(\pi_\Gamma^\theta, \lambda)$ is a faithful covariant representation of the C^*-dynamical system $(A, \Gamma \times G, \alpha^\Gamma)$. Now take $f \in L^1(\Gamma \times G, A)$ and assume there are $g \in L^1(\Gamma)$ and $h \in L^1(G, A)$ such that $f(\gamma, t) = g(\gamma)h(t)$ for all $(\gamma, t) \in \Gamma \times G$. Then

$$\begin{aligned}
\pi_\Gamma^\theta \rtimes \lambda(f) &= \int_{\Gamma \times G} \pi_\Gamma^\theta\big(f(\gamma,t)\big)\lambda_{(\gamma,s)}\, d(\gamma,s) \\
&= \int_{\Gamma \times G} \Big(g(\gamma) I_{L^2(\Gamma)} \otimes \pi^\theta\big(h(s)\big)\Big)(\lambda_\gamma^\Gamma \otimes \lambda_s^\theta)\, d(\gamma,s) \\
&= \Big(\int_\Gamma g(\gamma)\lambda_\gamma^\Gamma\, d\gamma\Big) \otimes \Big(\int_G \pi^\theta\big(h(s)\big)\lambda_s^\theta\, ds\Big) \\
&= \big(\lambda^\Gamma(g)\big) \otimes \big(\pi^\theta \rtimes \lambda^\theta(h)\big).
\end{aligned}$$

The result follows from the definition of each of the crossed products, and that of $\mathrm{vN}(\Gamma)$, since functions of the form of f above are dense in $L^1(\Gamma \times G, A)$ (see *e.g.* Ryan [51, Chapter 2]). $\square$

As in the result of De Cannière–Haagerup, Theorem 1.62, we will use the notion of multipliers which are bounded but not necessarily completely bounded, introduced in Definition 3.3.

Proposition 3.6. *Let (A, G, α) be a C^*-dynamical system, $(\theta, \mathcal{H}_\theta)$ a faithful representation of A, and $F : G \to \mathcal{CB}(A)$ a bounded function. The following are equivalent:*

i. F is a Herz–Schur θ-multiplier of (A, G, α);

ii. for every second-countable, locally compact group Γ, F^Γ is a θ-multiplier of $(A, \Gamma \times G, \alpha^\Gamma)$;

iii. $F^{\mathrm{SU}(2)}$ is a θ-multiplier of $(A, \mathrm{SU}(2) \times G, \alpha^{\mathrm{SU}(2)})$.

Moreover, if the conditions hold then $\|F\|_{\mathrm{HSw}} = \|F^{\mathrm{SU}(2)}\|_{\mathrm{w}}$.

Proof. (i)⇒(ii) Suppose that F is a Herz–Schur θ-multiplier of (A, G, α). Since the map Φ_F^θ on $A \rtimes_{\alpha,\theta}^{w^*} G$ is completely bounded and weak*-continuous, the map $\mathrm{id} \otimes \Phi_F^\theta$ is bounded and weak*-continuous on $\mathrm{vN}(\Gamma) \mathbin{\overline{\otimes}} (A \rtimes_{\alpha,\theta}^{w^*} G)$ and satisfies $\|\mathrm{id} \otimes \Phi_F^\theta\| \le \|\Phi_F^\theta\|_{\mathrm{cb}}$ [17, Lemma 1.5]. Moreover, using the notation of Lemma 3.5, and equations (3.2) and (3.3),

$$
\begin{aligned}
(\mathrm{id} \otimes \Phi_F^\theta)\big(\pi_\Gamma^\theta(a)\lambda_{(\gamma,t)}\big) &= (\mathrm{id} \otimes \Phi_F^\theta)\big(\lambda_\gamma^\Gamma \otimes \pi^\theta(a)\lambda_t^\theta\big) \\
&= \lambda_\gamma^\Gamma \otimes \pi^\theta\big(F(t)(a)\big)\lambda_t^\theta \\
&= \pi_\Gamma^\theta\big(F^\Gamma(\gamma,t)(a)\big)\lambda_{(\gamma,t)} \\
&= \Phi_{F^\Gamma}^\theta\big(\pi_\Gamma^\theta(a)\lambda_{(\gamma,t)}\big)
\end{aligned}
\tag{3.4}
$$

for all $a \in A,\ \gamma \in \Gamma,\ t \in G$. Thus, by Lemma 3.5, the map $\Phi_{F^\Gamma}^\theta$ defined by

$$
\Phi_{F^\Gamma}^\theta : \pi_\Gamma^\theta(a)\lambda_{(\gamma,t)} \mapsto \pi_\Gamma^\theta\big(F^\Gamma(\gamma,t)(a)\big)\lambda_{(\gamma,t)}
$$

extends to a bounded, weak*-continuous, map on $A \rtimes_{\alpha^\Gamma,\theta}^{w^*} (\Gamma \times G)$; *i.e.* F^Γ is a θ-multiplier of $(A, \Gamma \times G, \alpha^\Gamma)$.

(ii)⇒(iii) Trivial.

(iii)⇒(i) It is known — see, for example, Hall [29, Chapter I.4] — that for each $n \in \mathbb{N}$ the compact group SU(2) has exactly one irreducible representation of dimension n, so it follows from the Peter–Weyl Theorem (*e.g.* Folland [24, Theorem 5.12]) that $\mathrm{vN}(\mathrm{SU}(2)) \cong \oplus_{n\in\mathbb{N}} M_n$. Hence

$$
\mathrm{vN}(\mathrm{SU}(2)) \mathbin{\overline{\otimes}} (A \rtimes_{\alpha,\theta}^{w^*} G) \cong \oplus_{n\in\mathbb{N}}\big(M_n \otimes (A \rtimes_{\alpha,\theta}^{w^*} G)\big).
$$

By Lemma 3.5 $\Phi_{F^{\mathrm{SU}(2)}}^\theta$ is a bounded, weak*-continuous map on $\mathrm{vN}(\mathrm{SU}(2)) \mathbin{\overline{\otimes}} (A \rtimes_{\alpha,\theta}^{w^*} G)$, and (3.4) shows that $\Phi_{F^{\mathrm{SU}(2)}}^\theta = \mathrm{id} \otimes \Phi_F^\theta$. Hence, restricting to each component of the direct sum above, we obtain $\|\mathrm{id}_{M_n} \otimes \Phi_F^\theta\| \le \|\Phi_{F^{\mathrm{SU}(2)}}^\theta\|$ for all $n \in \mathbb{N}$; thus Φ_F^θ is completely bounded.

To prove the norm inequality first observe from the proof of (i)⇒(ii) that, for every locally compact group Γ, we have

$$\|F^{\Gamma}\|_{\mathrm{w}} = \|\Phi^{\theta}_{F^{\Gamma}}\| = \|\mathrm{id} \otimes \Phi^{\theta}_{F}\| \leq \|\Phi^{\theta}_{F}\|_{\mathrm{cb}} = \|F\|_{\mathrm{HSw}}.$$

On the other hand, from (iii)⇒(i),

$$\|F\|_{\mathrm{HSw}} = \|\Phi^{\theta}_{F}\|_{\mathrm{cb}} \leq \|\Phi^{\theta}_{F^{\mathrm{SU}(2)}}\| = \|F^{\mathrm{SU}(2)}\|_{\mathrm{w}}.$$

Hence $\|F\|_{\mathrm{HSw}} = \|F^{\mathrm{SU}(2)}\|_{\mathrm{w}}$. □

Let us review what has been achieved in this section. Definition 3.1 introduces Herz–Schur multipliers of a C^*-dynamical system; some basic properties of these functions are explored in Remarks 3.2, further investigation will be carried out in the next section where it is shown that they provide the correct framework for a Transference Theorem. A natural question about these new multipliers is whether we can characterise them similarly to Theorem 1.62, using the representation theory of vN(SU(2)). Lemma 3.5 shows that in order to make use of vN(SU(2)) we must have maps which act on the weak* closure of a reduced crossed product; since it is not clear which Herz–Schur multipliers of a C^*-dynamical system have suitable extensions to the weak* closure we introduce θ-multipliers in Definition 3.3; this definition is more straightforward than Definition 3.1. The gap between these definitions is explained by Remark 3.4 which shows that θ-multipliers, when viewed through a functional, act in the same way as Herz–Schur multipliers of the C^*-dynamical system. A further link between the two notions is given in Corollary 3.14.

3.2 Transference

Having established vector-valued versions of Schur and Herz–Schur multipliers the next goal is to connect these theories as in the Transference Theorem, Theorem 1.66. Recall that we consider a C^*-dynamical system (A, G, α), with the standing assumptions that A is a separable C^*-algebra and G is a second-countable, locally compact group.

The following will be used in Lemma 3.10; the statement may be of interest in its own right.

Lemma 3.7. *Let (A, G, α) be a C^*-dynamical system and $(\theta, \mathcal{H}_\theta)$ a faithful representation of A on a separable Hilbert space. Let $F : G \to \mathcal{CB}(A)$ be a bounded, pointwise-measurable function such that there exists $C > 0$ satisfying*

$$\|(\pi^\theta \rtimes \lambda^\theta)(F \cdot f)\| \le C\|(\pi^\theta \rtimes \lambda^\theta)(f)\|, \quad f \in L^1(G, A). \tag{3.5}$$

Fix $a \in A$ and $\omega \in \mathcal{B}(L^2(G, \mathcal{H}_\theta))_*$, and define

$$g_\omega : G \to \mathbb{C};\ g_\omega(s) := \left\langle \pi^\theta\big(F(s)(a)\big)\lambda_s^\theta, \omega \right\rangle, \quad s \in G.$$

Then g_ω coincides with an element of $\mathrm{MA}(G)$ *up to a null set.*

Proof. Let π_G^θ denote the representation of A on the Hilbert space $L^2(G, \mathcal{H}_\theta) \otimes L^2(G)$ given by $\pi_G^\theta(a) = \pi^\theta(a) \otimes I_{L^2(G)}$. The covariant pair $(\pi_G^\theta, \lambda^\theta \otimes \lambda^G)$ was considered in Lemma 3.5. We first show that the representation $\pi_G^\theta \rtimes (\lambda^\theta \otimes \lambda^G)$ is unitarily equivalent to a direct sum of copies of the representation $\pi^\theta \rtimes \lambda^\theta$. Define a unitary operator

$$U_1 : L^2(G) \otimes L^2(G) \to L^2(G, L^2(G));\ U_1(\xi \otimes \eta)(s) := \xi(s)\lambda_{s^{-1}}^G \eta, \quad \xi, \eta \in L^2(G).$$

It is clear that U_1 is surjective, so the calculation below and the density of elementary tensors in $L^2(G) \otimes L^2(G)$ imply that U_1 is unitary. Let $\xi_1, \xi_2, \eta_1, \eta_2 \in L^2(G)$, then

$$\begin{aligned}
\langle U_1(\xi_1 \otimes \eta_1), U_1(\xi_2 \otimes \eta_2)\rangle &= \int_G \langle U_1(\xi_1 \otimes \eta_1)(s), U_1(\xi_2 \otimes \eta_2)(s)\rangle \, ds \\
&= \int_G \left\langle \xi_1(s)\lambda_{s^{-1}}^G \eta_1, \xi_2(s)\lambda_{s^{-1}}^G \eta_2 \right\rangle \, ds \\
&= \int_G \xi_1(s)\overline{\xi_2(s)}\, \langle \eta_1, \eta_2\rangle \, ds \\
&= \langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle .
\end{aligned}$$

Let $(\eta_i)_{i \in \mathbb{N}}$ be an orthonormal basis for $L^2(G)$. Define another unitary operator

$$U_2 : L^2(G, L^2(G)) \to \oplus_{i \in I} L^2(G);\ (U_2 f)_i(s) := \langle f(s), \eta_i\rangle ,$$

for all $i \in I,\ s \in G,\ f \in L^2(G, L^2(G))$; indeed, U_2 implements the standard unitary equivalence of $L^2(G, L^2(G))$ and $\oplus_{i\in I} L^2(G)$. Let $U := U_2 U_1$, which is a unitary operator from $L^2(G) \otimes L^2(G)$ to $\oplus_{i\in I} L^2(G)$. For an operator T on a Hilbert space $\mathcal{H}$ we will denote by T^∞ the amplification of T, acting on $\oplus_{i\in I}\mathcal{H}$, given by

$$T^\infty\big((\eta_i)_{i\in I}\big) := (T\eta_i)_{i\in I}, \quad (\eta_i)_{i\in I} \in \oplus_{i\in I}\mathcal{H}. \tag{3.6}$$

Let $\xi_1 \in \mathcal{H}_\theta,\ \xi_2, \xi_3 \in L^2(G),\ a \in A$ and $t \in G$. We use ζ_i to denote the function $\zeta_i(s) := \xi_2(s) \left\langle \xi_3, \lambda_s^G \eta_i \right\rangle$ $(s \in G)$. For almost all $s \in G$ we have

$$\begin{aligned}
&(I_{\mathcal{H}_\theta} \otimes U)\pi_G^\theta(a)(\xi_1 \otimes \lambda_t^G \xi_2 \otimes \lambda_t^G \xi_3)(s) \\
&\quad = \Big(\theta\big(\alpha_{s^{-1}}(a)\big)\xi_1 (\lambda_t^G \xi_2)(s) \left\langle \lambda_{s^{-1}}^G \lambda_t^G \xi_3, \eta_i \right\rangle\Big)_{i\in I} \\
&\quad = \Big(\theta\big(\alpha_{s^{-1}}(a)\big)\xi_1 (\lambda_t^G \xi_2)(s) \left\langle \xi_3, \lambda_{t^{-1}s}^G \eta_i \right\rangle\Big)_{i\in I} \\
&\quad = \Big(\theta\big(\alpha_{s^{-1}}(a)\big)\xi_1 \lambda_t^G \zeta_i(s)\Big)_{i\in I} \\
&\quad = \pi^\theta(a)^\infty \big(\lambda_t^\theta(\xi_1 \otimes \zeta_i)(s)\big)_{i\in I} \\
&\quad = \pi^\theta(a)^\infty \big((\lambda_t^\theta \otimes I_{L^2(G)})(I_{\mathcal{H}_\theta} \otimes U)(\xi_1 \otimes \xi_2 \otimes \xi_3)\big)(s).
\end{aligned}$$

Since sums of elementary tensors are dense in $L^2(G, \mathcal{H}_\theta) \otimes L^2(G)$ it follows that

$$(I_{\mathcal{H}_\theta} \otimes U)\pi_G^\theta(a)(\lambda_t^\theta \otimes \lambda_t^G) = \big(\pi^\theta(a)\lambda_t^\theta\big)^\infty (I_{\mathcal{H}_\theta} \otimes U)$$

for all $a \in A,\ t \in G$. Thus, for $f \in L^1(G, A)$,

$$\begin{aligned}
(I_{\mathcal{H}_\theta} \otimes U)\big(\pi_G^\theta \rtimes (\lambda^\theta \otimes \lambda^G)(f)\big) &= (I_{\mathcal{H}_\theta} \otimes U) \int_G \pi_G^\theta\big(f(s)\big)(\lambda_s^\theta \otimes \lambda_s^G)\, ds \\
&= \int_G (I_{\mathcal{H}_\theta} \otimes U)\pi_G^\theta\big(f(s)\big)(\lambda_s^\theta \otimes \lambda_s^G)\, ds \\
&= \int_G \Big(\pi^\theta\big(f(s)\big)\lambda_t^\theta\Big)^\infty (I_{\mathcal{H}_\theta} \otimes U)\, ds \\
&= \big(\pi^\theta \rtimes \lambda^\theta(f)\big)^\infty (I_{\mathcal{H}_\theta} \otimes U),
\end{aligned}$$

which is the unitary equivalence claimed.

Now fix $\omega \in \mathcal{B}(L^2(G,\mathcal{H}))_*$, take $v \in B_\lambda(G)$ and let w denote the linear functional on $\{\lambda^G(f) : f \in L^1(G)\}$ given by

$$w\big(\lambda^G(f)\big) = \int_G f(s) g_\omega(s) v(s)\, ds.$$

Let $f \in L^1(G)$, $a \in A$, and define $\tilde{f}(s) := f(s)a$ $(s \in G)$; clearly $\tilde{f} \in L^1(G, A)$. Fix $\xi, \eta \in L^2(G)$ and let $\omega_{\xi,\eta}$ be the associated vector functional on $\mathcal{B}(L^2(G))$. Then

$$\begin{aligned}
\big\langle \lambda^G(fg_\omega)\xi, \eta \big\rangle &= \int_G f(s) \left\langle \pi^\theta\big(F(s)(a)\big), \omega \right\rangle \left\langle \lambda_s^G \xi, \eta \right\rangle\, ds \\
&= \left\langle \int_G \Big(\pi^\theta\big(F(s)(\tilde{f}(s))\big) \otimes I_{L^2(G)}\Big)(\lambda_s^\theta \otimes \lambda_s^G)\, ds, \omega \otimes \omega_{\xi,\eta} \right\rangle \\
&= \left\langle \int_G \Big(\pi_G^\theta\big(F(s)(\tilde{f}(s))\big)\Big)(\lambda_s^\theta \otimes \lambda_s^G)\, ds, \omega \otimes \omega_{\xi,\eta} \right\rangle \\
&= \left\langle \pi_G^\theta \rtimes (\lambda^\theta \otimes \lambda^G)(F \cdot \tilde{f}), \omega \otimes \omega_{\xi,\eta} \right\rangle.
\end{aligned}$$

Now since $\pi_G^\theta \rtimes (\lambda^\theta \otimes \lambda^G)$ is unitarily equivalent to a direct sum of copies of $\pi^\theta \rtimes \lambda^\theta$ we have

$$\|\lambda^G(fg_\omega)\| \le \|\omega\| \|\pi_G^\theta \rtimes (\lambda^\theta \otimes \lambda^G)(F \cdot \tilde{f})\| = \|\omega\| \|(\pi^\theta \rtimes \lambda^\theta)(F \cdot \tilde{f})\|.$$

As $(\pi^\theta \rtimes \lambda^\theta)(F \cdot \tilde{f}) = \int_G f(s)\pi^\theta(F(s)(a))\lambda_s^\theta\, ds$ we have, by (3.5),

$$\begin{aligned}
\|w\big(\lambda^G(f)\big)\| &\le \|v\|_\lambda \|\lambda^G(fg_\omega)\| \\
&\le C\|\omega\| \|v\|_\lambda \|(\pi^\theta \rtimes \lambda^\theta)(\tilde{f})\| \\
&\le C\|\omega\| \|v\|_\lambda \left\| \pi^\theta(a) \int f(s)\lambda_s^\theta\, ds \right\| \\
&\le C\|\omega\| \|v\|_\lambda \|\pi^\theta(a)\| \left\| \int f(s)\lambda_s^\theta\, ds \right\| \\
&\le C\|\omega\| \|v\|_\lambda \|\pi^\theta(a)\| \|\lambda^\theta(f)\| \\
&\le C\|\omega\| \|v\|_\lambda \|\pi^\theta(a)\| \|\lambda^G(f)\|.
\end{aligned}$$

It follows that w extends to a bounded linear functional on $C_\lambda^*(G)$; since $C_\lambda^*(G)^* = B_\lambda(G)$ there exists $z \in B_\lambda(G)$ such that $w(\lambda^G(f)) = \int_G f(s)z(s)\, ds$ $(f \in L^1(G))$. It follows that $z = vg_\omega$ almost everywhere.

We have shown that for all $v \in B_\lambda(G)$ there exists $z \in B_\lambda(G)$ such that $z = vg_\omega$ almost everywhere. Since every element of $B_\lambda(G)$ is continuous g_ω must be equal almost everywhere to a continuous function g'_ω. Now we have that for all $v \in B_\lambda(G)$ vg'_ω is equal almost everywhere to an element $z \in B_\lambda(G)$; since vg'_ω is continuous it follows that $vg'_\omega = z$ everywhere; thus $vg'_\omega \in B_\lambda(G)$ for all $v \in B_\lambda(G)$. By De Cannière–Haagerup, Theorem 1.61, $g'_\omega \in \mathrm{M}A(G)$, so g_ω is equal almost everywhere to the function $g'_\omega \in \mathrm{M}A(G)$. □

We observe that in the separable case the null set of Lemma 3.7 can be chosen independently of the functional ω.

Remark 3.8. Let (A, G, α) be a C^*-dynamical system and $(\theta, \mathcal{H}_\theta)$ a faithful representation of A. Let $F : G \to \mathcal{CB}(A)$ be a pointwise-measurable function satisfying condition (3.5) and fix $a \in A$. For $\omega \in \mathcal{B}(L^2(G, \mathcal{H}_\theta))_*$ define

$$g_\omega : G \to \mathbb{C};\ g_\omega(s) := \left\langle \pi^\theta\big(F(s)(a)\big)\lambda^\theta_s, \omega \right\rangle, \quad s \in G,$$

and find, by Lemma 3.7, $b_\omega \in \mathrm{M}A(G)$ such that $b_\omega = g_\omega$ almost everywhere. If G is second-countable and $\mathcal{H}_\theta$ is separable then there exists a null set $M \subseteq G$ such that, for any $\omega \in \mathcal{B}(L^2(G, \mathcal{H}_\theta))_*$, $g_\omega(t) = b_\omega(t)$ for all $t \in G \setminus M$.

Proof. By Remark 1.42 the fact that G is second-countable and $\mathcal{H}_\theta$ is separable implies $L^2(G, \mathcal{H}_\theta)$ is separable, so that $\mathcal{B}(L^2(G, \mathcal{H}_\theta))_*$ is separable [19, Chapter I.3 Ex. 4]. Let $\{\omega_n : n \in \mathbb{N}\} \subseteq \mathcal{B}(L^2(G, \mathcal{H}_\theta))_*$ be a dense subset. For each $n \in \mathbb{N}$ use Lemma 3.7 to find $b_{\omega_n} \in \mathrm{M}A(G)$, and a null set $M_n \subseteq G$, such that $g_{\omega_n}(t) = b_{\omega_n}(t)$ for all $t \in G \setminus M_n$. Set $M := \cup_{n\in\mathbb{N}} M_n$, which is a null subset of G. For $\omega \in \mathcal{B}(L^2(G, \mathcal{H}_\theta))_*$ let $(\omega_{n_k})_{k\in\mathbb{N}}$ be a subset converging to ω in norm. Then $(b_{\omega_{n_k}})_{k\in\mathbb{N}}$ is a Cauchy sequence of bounded, continuous, functions. Indeed, let $K = \|\omega\| \sup_{s\in G} \|F(s)\|$ and take $\epsilon > 0$; since the sequence $(\omega_{n_k})_{k\in\mathbb{N}}$ converges there exists $L \in \mathbb{N}$ such that for $k, l > L$ we have

$$\left|b_{\omega_{n_k}}(t) - b_{\omega_{n_l}}(t)\right| = \left|g_{\omega_{n_k}}(t) - g_{\omega_{n_l}}(t)\right| \le K\|\omega_{n_k} - \omega_{n_l}\| < K\epsilon$$

for all $t \in G \setminus M$. As each $b_{\omega_{n_k}}$ is continuous, $\left|b_{\omega_{n_k}}(t) - b_{\omega_{n_l}}(t)\right| < K\epsilon$ for all $t \in G$. Thus the sequence $(b_{\omega_{n_k}})_{k\in\mathbb{N}}$ converges to a continuous function b. We also have $b_{\omega_{n_k}}(t) \to$

$g_\omega(t)$ for all $t \in G \setminus M$. We conclude that $g_\omega(t) = b(t)$ for all $t \in G \setminus M$. As b_ω and b are continuous, and $b_\omega = g_\omega$ almost everywhere, we have $b = b_\omega$ as required. □

The following notion will be used several times in the results below.

Definition 3.9. Let G be a locally compact group and $t \in G$. A *Dirac family at* t is a net $(f_U)_U \subseteq L^1(G)$ of non-negative functions, indexed by the set of all open neighbourhoods of t with compact closure, such that $\operatorname{supp} f_U \subseteq U$ and $\|f_U\|_1 = 1$.

Lemma 3.10. *Let $(f_U)_U$ be a Dirac family at the point $t \in G$, (ρ, τ) a covariant representation of the C^*-dynamical system (A, G, α) on a Hilbert space $\mathcal{H}_{\rho,\tau}$, and $a \in A$. Then*

$$\rho \rtimes \tau(f_U \otimes a) \xrightarrow{U} \rho(a)\tau_t \tag{3.7}$$

in the weak operator topology. Moreover, if F is a Herz–Schur (A, G, α)-multiplier and $(\theta, \mathcal{H}_\theta)$ a faithful representation of A on a separable Hilbert space then there exists a null set $M \subseteq G$ such that if $t \in G \setminus M$ and $(f_U)_U$ is a Dirac family at t then, for every $a \in A$,

$$S_F\big((\pi^\theta \rtimes \lambda^\theta)(f_U \otimes a)\big) \to \pi^\theta\big(F(t)(a)\big)\lambda^\theta_t \tag{3.8}$$

in the weak topology.*

Proof. Fix $a \in A$ and note that the map $s \mapsto \rho(a)\tau_s$ is continuous. Take $\xi, \eta \in \mathcal{H}_{\rho,\tau}$ and let

$$C_U := \sup_{s \in U} |\langle \rho(a)\tau_s\xi, \eta\rangle - \langle \rho(a)\tau_t\xi, \eta\rangle|\,;$$

since $s \mapsto \rho(a)\tau_s$ is continuous it follows that $C_U \to 0$. We have

$$\begin{aligned}
&|\langle \rho \rtimes \tau(f_U \otimes a)\xi, \eta\rangle - \langle \rho(a)\tau_t\xi, \eta\rangle| \\
&\qquad = \left| \int_G \langle \rho(f_U(s)a)\tau_s\xi, \eta\rangle \, ds - \int_G \langle f_U(s)\rho(a)\tau_t\xi, \eta\rangle \, ds \right| \\
&\qquad \le \int_G f_U(s)\, |\langle \rho(a)\tau_s\xi, \eta\rangle - \langle \rho(a)\tau_t\xi, \eta\rangle| \, ds \\
&\qquad \le C_U \int_G f_U(s)\, ds \to 0.
\end{aligned}$$

Equation (3.7) follows.

For the second statement fix $a \in A$. By Lemma 3.7 and Remark 3.8 there is a null set $M \subseteq G$ such that, for any $\xi, \eta \in L^2(G, \mathcal{H}_\theta)$, there is a continuous function $b_{\xi,\eta} : G \to \mathbb{C}$ satisfying

$$\Big\langle \pi^\theta\big(F(t)(a)\big)\lambda^\theta_t \xi, \eta \Big\rangle = b_{\xi,\eta}(t), \quad t \in G \setminus M.$$

Fix $\xi, \eta \in L^2(G, \mathcal{H}_\theta)$ and let $t \in G \setminus M$ and $(f_U)_U$ be a Dirac family at t; define

$$D_U := \sup_{s \in U} \big| b_{\xi,\eta}(s) - b_{\xi,\eta}(t) \big|.$$

Since $b_{\xi,\eta}$ is continuous $D_U \to 0$. We have

$$\begin{aligned}
&\Big| \Big\langle S_F\big(\pi^\theta \rtimes \lambda^\theta (f_U \otimes a)\big)\xi, \eta \Big\rangle - \Big\langle \pi^\theta\big(F(t)(a)\big)\lambda^\theta_t \xi, \eta \Big\rangle \Big| \\
&= \Big| \int_G \Big\langle \pi^\theta\big(F(s)(a)\big) f_U(s) \lambda^\theta_s \xi, \eta \Big\rangle \, ds - \int_G \Big\langle f_U(s) \pi^\theta\big(F(t)(a)\big)\lambda^\theta_t \xi, \eta \Big\rangle \, ds \Big| \\
&\leq \int_G f_U(s) \Big| \Big\langle \pi^\theta\big(F(s)(a)\big)\lambda^\theta_s \xi, \eta \Big\rangle - \Big\langle \pi^\theta\big(F(t)(a)\big)\lambda^\theta_t \xi, \eta \Big\rangle \Big| \, ds \\
&= \int_G f_U(s) \big| b_{\xi,\eta}(s) - b_{\xi,\eta}(t) \big| \, ds \\
&\leq D_U \int_G f_U(s) \, ds \\
&= D_U \to 0.
\end{aligned}$$

Equation (3.8) follows, since the weak* topology coincides with the weak operator topology on bounded sets (*e.g.* Dixmier [18, A.1]). □

If $\varphi : G \times G \to \mathcal{CB}(A)$ is a bounded, pointwise-measurable function then define

$$\mathcal{T}(\varphi) : G \times G \to \mathcal{CB}(A);\ \mathcal{T}(\varphi)(s,t)(a) := \alpha_t\Big(\varphi(s,t)\big(\alpha_{t^{-1}}(a)\big)\Big), \quad a \in A,\ (s,t) \in G \times G,$$

which is bounded and pointwise-measurable because φ is so and α is point-norm continuous. Note that the inverse is given by

$$\mathcal{T}^{-1}(\varphi) : G \times G \to \mathcal{CB}(A);\ \mathcal{T}^{-1}(\varphi)(s,t)(a) := \alpha_{t^{-1}}\Big(\varphi(s,t)\big(\alpha_t(a)\big)\Big),$$

for all $a \in A$, $(s,t) \in G \times G$, and that $\mathcal{T}^{-1}(\varphi)$ is again bounded and pointwise-measurable if φ is so. Recall that for a (possibly vector-valued) function $\phi : G \to X$ we define

$$N(\phi) : G \times G \to X;\; N(\phi)(s,t) := \phi(ts^{-1}), \quad (s,t) \in G \times G,$$

which is bounded and pointwise-measurable if ϕ is so, because the map $(s,t) \mapsto ts^{-1}$ is measurable from $G \times G \to G$. For a bounded, pointwise-measurable function $F : G \to \mathcal{CB}(A)$ we let

$$\mathcal{N}(F) : G \times G \to \mathcal{CB}(A);\; \mathcal{N}(F) := \mathcal{T}^{-1}\big(N(F)\big),$$

so that

$$\mathcal{N}(F)(s,t)(a) = \alpha_{t^{-1}}\Big(F(ts^{-1})\big(\alpha_t(a)\big)\Big), \quad (s,t) \in G \times G,\; a \in A.$$

The above comments imply that $\mathcal{N}(F)$ is bounded and pointwise-measurable if F is so. The following result is a vector-valued version of Theorem 1.66, which describes Herz–Schur multipliers in terms of Schur multipliers.

Theorem 3.11. *Let $F : G \to \mathcal{CB}(A)$ be a bounded, pointwise-measurable function. The following are equivalent:*

i. F is a Herz–Schur (A, G, α)-multiplier;

ii. $\mathcal{N}(F)$ is a Schur A-multiplier.

Moreover, if the conditions hold then $\|F\|_{\mathrm{HS}} = \|\mathcal{N}(F)\|_{\mathfrak{S}}$.

Proof. (i)⇒(ii) Suppose that F is a Herz–Schur (A, G, α)-multiplier and let $(\theta, \mathcal{H}_\theta)$ be a faithful representation of A on a separable Hilbert space. By the Haagerup–Paulsen–Wittstock Theorem, Theorem 1.18, there exist a representation $(\rho, \mathcal{H}_\rho)$ of $A \rtimes_{\alpha,\theta} G$ on a separable Hilbert space, and operators $V, W : L^2(G, \mathcal{H}_\theta) \to \mathcal{H}_\rho$, such that

$$S_F(T) = W^* \rho(T) V, \quad T \in A \rtimes_{\alpha,\theta} G, \tag{3.9}$$

and $\|S_F\|_{\mathrm{cb}} = \|V\|\|W\|$. Consider the full crossed product $A \rtimes_\alpha G$ associated to (A, G, α). Let $q : A \rtimes_\alpha G \to A \rtimes_{\alpha,\theta} G$ be the quotient map and define a representation $\tilde{\rho}$ of $A \rtimes_\alpha G$ on $\mathcal{H}_\rho$ by $\tilde{\rho} = \rho \circ q$. By Proposition 1.91 there exist a representation

$(\rho_A, \mathcal{H}_\rho)$ of A, and a strongly continuous unitary representation ρ_G of G on $\mathcal{H}_\rho$, such that $\tilde{\rho} = \rho_A \rtimes \rho_G$. We have

$$\tilde{\rho}(f) = \int_G \rho_A\big(f(s)\big)\rho_G(s)\,ds, \quad f \in L^1(G, A).$$

Applying (3.9) with $T = \pi^\theta \rtimes \lambda^\theta(f)$ ($f \in L^1(G, A) \subseteq A \rtimes_\alpha G$) we have

$$\begin{aligned}
\int_G \pi^\theta\Big(F(s)\big(f(s)\big)\Big)\lambda_s^\theta\,ds &= S_F\big(\pi^\theta \rtimes \lambda^\theta(f)\big) \\
&= W^*\rho\big(\pi^\theta \rtimes \lambda^\theta(f)\big)V \\
&= W^*\rho \circ q(f)V \\
&= W^*\tilde{\rho}(f)V \\
&= W^*\left(\int_G \rho_A\big(f(s)\big)\rho_G(s)\,ds\right)V.
\end{aligned} \tag{3.10}$$

If $a \in A$ and $(f_U)_U$ is a Dirac family at $t \in G$ then, by Lemma 3.10,

$$\int_G \rho_A\big((f_U \otimes a)(s)\big)\rho_G(s)\,ds \xrightarrow{U} \rho_A(a)\rho_G(t) \tag{3.11}$$

in the weak operator topology. Taking $f = f_U \otimes a$ in (3.10), applying (3.11) and Lemma 3.10, we obtain a null set $M \subseteq G$ such that

$$\pi^\theta\big(F(t)(a)\big)\lambda_t^\theta = W^*\rho_A(a)\rho_G(t)V \tag{3.12}$$

for all $t \in G \setminus M$. For $s \in G$ define $\mathcal{V}(s), \mathcal{W}(s) \in \mathcal{B}(L^2(G, \mathcal{H}_\theta), \mathcal{H}_\rho)$ by

$$\mathcal{V}(s) := \rho_G(s^{-1})V\lambda_s^\theta, \quad \mathcal{W}(s) := \rho_G(s^{-1})W\lambda_s^\theta.$$

For every $\xi \in L^2(G, \mathcal{H}_\theta)$ the functions $s \mapsto \mathcal{V}(s)\xi$ and $s \mapsto \mathcal{W}(s)\xi$ are continuous, and

$$\operatorname*{ess\,sup}_{s\in G} \|\mathcal{V}(s)\| = \|V\| \;\text{ and }\; \operatorname*{ess\,sup}_{s\in G} \|\mathcal{W}(s)\| = \|W\|, \tag{3.13}$$

so $\mathcal{V}, \mathcal{W} \in L^\infty(G, \mathcal{B}(L^2(G, \mathcal{H}_\theta), \mathcal{H}_\rho))$. By (3.12) we obtain

$$\begin{aligned}
\mathcal{W}(t)^*\rho_A(a)\mathcal{V}(s) &= \lambda^\theta_{t^{-1}} W^* \rho_G(t)\rho_A(a)\rho_G(s^{-1})V\lambda^\theta_s \\
&= \lambda^\theta_{t^{-1}} W^* \rho_A\big(\alpha_t(a)\big)\rho_G(ts^{-1})V\lambda^\theta_s \\
&= \lambda^\theta_{t^{-1}} \pi^\theta\Big(F(ts^{-1})\big(\alpha_t(a)\big)\Big)\lambda^\theta_{ts^{-1}}\lambda^\theta_s \\
&= \pi^\theta\Big(\alpha_{t^{-1}}\big(F(ts^{-1})\big(\alpha_t(a)\big)\big)\Big) \\
&= \pi^\theta\big(\mathcal{N}(F)(s,t)(a)\big)
\end{aligned}$$

for all $a \in A$ and all $(s,t) \in G \times G$ with $ts^{-1} \in G \setminus M$. It follows from the properties of Haar measure that $\{(s,t) \in G \times G : ts^{-1} \in M\}$ is a null set for the product measure; therefore, by Theorem 2.9, $\mathcal{N}(F)^{\pi^\theta}$ is a Schur $\pi^\theta(A)$-multiplier. By Proposition 2.6 it follows that $\mathcal{N}(F)$ is a Schur A-multiplier. It follows from (3.13) that

$$\|\mathcal{N}(F)\|_{\mathfrak{S}} \le \|V\|\|W\| = \|F\|_{\mathrm{HS}}. \tag{3.14}$$

(ii)⇒(i) Let $(\theta, \mathcal{H}_\theta)$ be a faithful representation of A on a separable Hilbert space. Suppose that $\mathcal{N}(F)^\theta$ is a Schur A-multiplier. Let $f \in C_c(G, A)$. For any $\xi \in L^2(G, \mathcal{H}_\theta)$ and almost all $t \in G$ we have

$$\begin{aligned}
\pi^\theta \rtimes \lambda^\theta(f)\xi(t) &= \int_G \pi^\theta\big(f(s)\big)\lambda^\theta_s\xi(t)\,ds \\
&= \int_G \theta\Big(\alpha_{t^{-1}}\big(f(s)\big)\Big)\big(\lambda^\theta_s\xi(t)\big)\,ds \\
&= \int_G \theta\Big(\alpha_{t^{-1}}\big(f(s)\big)\Big)\big(\xi(s^{-1}t)\big)\,ds \\
&= \int_G \Delta(r)^{-1}\theta\Big(\alpha_{t^{-1}}\big(f(tr^{-1})\big)\Big)\big(\xi(r)\big)\,dr.
\end{aligned}$$

Fix a compact set $K \subseteq G$. Then the function

$$h_K : (t,r) \mapsto \chi_{K\times K}(t,r)\Delta(r)^{-1}\theta\Big(\alpha_{t^{-1}}\big(f(tr^{-1})\big)\Big) \tag{3.15}$$

is an element of $L^2(G \times G, \theta(A))$. Indeed,

$$\|h_K\|_2^2 = \int_{G\times G} \|h_K(t,r)\|^2\,d(r,t) \le \int_{K\times K} \Delta(r^{-1})^2\|f(tr^{-1})\|^2\,d(r,t),$$

which is finite. For any $\xi \in L^2(G, \mathcal{H}_\theta)$ and $s \in G$ we have

$$\begin{aligned}(T_{h_K}\xi)(s) &= \int_G h_K(s,t)\big(\xi(t)\big)\,dt \\ &= \int_G \chi_{K\times K}(s,t)\Delta(t)^{-1}\theta\Big(\alpha_{s^{-1}}\big(f(st^{-1})\big)\Big)\big(\xi(t)\big)\,dt\end{aligned}$$

and

$$\begin{aligned}\Big((M_{\chi_K}\otimes I_{\mathcal{H}_\theta})\big(\pi^\theta \rtimes \lambda^\theta(f)\big)(M_{\chi_K}\otimes I_{\mathcal{H}_\theta})\Big)\xi(s) &= \chi_K(s)\int_G \pi^\theta\big(f(r)\big)\big(\lambda^\theta_r(\chi_K\xi)\big)(s)\,dr \\ &= \chi_K(s)\int_G \theta\Big(\alpha_{s^{-1}}\big(f(r)\big)\Big)\big((\chi_K\xi)(r^{-1}s)\big)\,dr \\ &= \chi_K(s)\int_G \chi_K(t)\Delta(t)^{-1}\theta\Big(\alpha_{s^{-1}}\big(f(st^{-1})\big)\Big)\big(\xi(t)\big)\,dt.\end{aligned}$$

Thus

$$T_{h_K} = (M_{\chi_K}\otimes I_{\mathcal{H}_\theta})(\pi^\theta \rtimes \lambda^\theta)(f)(M_{\chi_K}\otimes I_{\mathcal{H}_\theta}). \tag{3.16}$$

Now

$$\begin{aligned}\mathcal{N}(F)^\theta\cdot h_K(t,r) &= \chi_{K\times K}(t,r)\Delta(r)^{-1}\theta\Big(\alpha_{t^{-1}}\big(F(tr^{-1})\big(\alpha_t\big(\alpha_{t^{-1}}(f(tr^{-1}))\big)\big)\big)\Big) \\ &= \chi_{K\times K}(t,r)\Delta(r)^{-1}\theta\Big(\alpha_{t^{-1}}\big(F(tr^{-1})\big(f(tr^{-1})\big)\big)\Big).\end{aligned}$$

Let $\xi, \eta \in L^2(G, \mathcal{H}_\theta)$ have compact support, recall that G is assumed to be second-countable, and choose by Proposition 1.3 an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of G such that $G = \cup_{n\in\mathbb{N}} K_n$. Then

$$\Big\langle S_{\mathcal{N}(F)^\theta}(T_{h_{K_n}})\xi,\eta\Big\rangle = \int_{K_n\times K_n}\Delta(r)^{-1}\Big\langle\theta\Big(\alpha_{t^{-1}}\big(F(tr^{-1})\big(f(tr^{-1})\big)\big)\Big)\xi(r),\eta(t)\Big\rangle\,d(r,t). \tag{3.17}$$

We have

$$\begin{aligned}&\Big|\Delta(r)^{-1}\Big\langle\theta\Big(\alpha_{t^{-1}}\big(F(tr^{-1})\big(f(tr^{-1})\big)\big)\Big)\xi(r),\eta(t)\Big\rangle\Big| \\ &\qquad\le \Delta(r)^{-1}\sup_{s\in G}\|F(s)\|\|f(tr^{-1})\|\|\xi(r)\|\|\eta(t)\|,\end{aligned}$$

and

$$\int_{G\times G}\Delta(r)^{-1}\|f(tr^{-1})\|\|\xi(r)\|\|\eta(t)\|\,d(r,t) = \left\langle f' * \xi', \eta'\right\rangle,$$

where $f', \xi', \eta' : G \to \mathbb{R}$ are the functions $f'(s) := \|f(s)\|$, $\xi'(s) := \|\xi(s)\|$, $\eta'(s) :=$

$\|\eta(s)\|$ $(s \in G)$. The Lebesgue Dominated Convergence Theorem (see *e.g.* Cohn [13, Theorem 2.4.5]) implies that the right side of (3.17) converges to

$$\int_{G\times G} \Delta(r)^{-1} \left\langle \theta\Big(\alpha_{t^{-1}}\big(F(tr^{-1})\big(f(tr^{-1})\big)\big)\Big)\xi(r), \eta(t)\right\rangle \, d(r,t) = \left\langle \pi^\theta \rtimes \lambda^\theta(F\cdot f)\xi, \eta\right\rangle.$$

Thus

$$\lim_{n\to\infty} \left\langle S_{\mathcal{N}(F)^\theta}(T_{h_{K_n}})\xi, \eta\right\rangle = \left\langle \pi^\theta \rtimes \lambda^\theta(F\cdot f)\xi, \eta\right\rangle. \tag{3.18}$$

By (3.16) $T_{h_{K_n}} \xrightarrow{n} \pi^\theta \rtimes \lambda^\theta(f)$ in the weak* topology, since $\chi_{K_n} \to I_{L^2(G)}$ in the weak* topology. It follows that

$$\begin{aligned}
\left|\left\langle \pi^\theta \rtimes \lambda^\theta(F\cdot f)\xi, \eta\right\rangle\right| &\le \limsup_{n\in\mathbb{N}} \left|\left\langle S_{\mathcal{N}(F)^\theta}(T_{h_{K_n}})\xi, \eta\right\rangle\right| \\
&\le \|\mathcal{N}(F)\|_{\mathfrak{S}}\|\xi\|\|\eta\| \limsup_{n\in\mathbb{N}} \|T_{h_{K_n}}\| \\
&\le \|\mathcal{N}(F)\|_{\mathfrak{S}}\|\xi\|\|\eta\|\|\pi^\theta \rtimes \lambda^\theta(f)\|.
\end{aligned}$$

So

$$\|\pi^\theta \rtimes \lambda^\theta(F\cdot f)\| \le \|\mathcal{N}(F)\|_{\mathfrak{S}}\|\pi^\theta \rtimes \lambda^\theta(f)\|;$$

hence S_F is a bounded map.

The proof that S_F is completely bounded amounts to identifying $M_n(\mathcal{S}_2(X,Y;A))$ with $\mathcal{S}_2(X,Y;M_n(A))$ and running the same argument as above. Take $m \in \mathbb{N}$ and $f_{i,j} \in L^1(G,A)$ for $1 \le i,j \le m$. To each $\pi^\theta \rtimes \lambda^\theta(f_{i,j})$, and each compact set $K \subseteq G$, associate $T_{h_K^{i,j}}$ as in (3.16). Letting $\xi = (\xi_i)_{i=1}^m, \eta = (\eta_i)_{i=1}^m \in L^2(G,\mathcal{H})^n$ we have, from (3.17), that

$$\begin{aligned}
\left\langle S^{(m)}_{\mathcal{N}(F)^\theta}\left(\left(T_{h^{i,j}_{K_n}}\right)_{i,j=1}^m\right)\xi, \eta\right\rangle &= \sum_{i,j=1}^m \left\langle S_{\mathcal{N}(F)^\theta}(T_{h^{i,j}_{K_n}})\xi_j, \eta_i\right\rangle \\
&= \int_{K_n\times K_n} \sum_{i,j=1}^m \Delta(r)^{-1} \left\langle \theta\Big(\alpha_{t^{-1}}\big(F(tr^{-1})\big(f_{i,j}(tr^{-1})\big)\big)\Big)\xi_j(r), \eta_i(t)\right\rangle \, d(r,t).
\end{aligned}$$

Applying the Lebesgue Dominated Convergence Theorem as above yields

$$\lim_{n\to\infty} \left\langle S^{(m)}_{\mathcal{N}(F)^\theta}\left(\left(T_{h^{i,j}_{K_n}}\right)_{i,j=1}^m\right)\xi, \eta\right\rangle = \left\langle \left(\pi^\theta \rtimes \lambda^\theta(F\cdot f_{i,j})\right)_{i,j=1}^m \xi, \eta\right\rangle.$$

Thus we have

$$\begin{aligned}\left|\left\langle \left(\pi^\theta \rtimes \lambda^\theta(F\cdot f_{i,j})\right)_{i,j=1}^m \xi, \eta\right\rangle\right| &\le \limsup_{n\in\mathbb{N}} \left|\left\langle S^{(m)}_{\mathcal{N}(F)^\theta}\left(\left(T_{h^{i,j}_{K_n}}\right)_{i,j=1}^m\right)\xi,\eta\right\rangle\right| \\ &\le \|\mathcal{N}(F)\|_{\mathfrak{S}}\|\xi\|\|\eta\| \limsup_{n\in\mathbb{N}} \left\|\left(T_{h^{i,j}_{K_n}}\right)_{i,j=1}^m\right\| \\ &\le \|\mathcal{N}(F)\|_{\mathfrak{S}}\|\xi\|\|\eta\| \left\|\left(\pi^\theta \rtimes \lambda^\theta(f_{i,j})\right)_{i,j=1}^m\right\|,\end{aligned}$$

and therefore

$$\left\|\left(\pi^\theta \rtimes \lambda^\theta(F\cdot f_{i,j})\right)_{i,j=1}^m\right\| \le \|\mathcal{N}(F)\|_{\mathfrak{S}} \left\|\left(\pi^\theta \rtimes \lambda^\theta(f_{i,j})\right)_{i,j=1}^m\right\|,$$

so S_F is completely bounded and $\|S_F\|_{\mathrm{cb}} \le \|\mathcal{N}(F)^\theta\|_{\mathfrak{S}}$. By Remark 3.2(ii) F is a Herz–Schur (A, G, α)-multiplier and

$$\|F\|_{\mathrm{HS}} \le \|\mathcal{N}(F)\|_{\mathfrak{S}}.$$

Together with (3.14) this implies $\|F\|_{\mathrm{HS}} = \|\mathcal{N}(F)\|_{\mathfrak{S}}$. □

Remark 3.12. In the case $A = \mathbb{C}$ Theorem 3.11 reduces to the classical transference result, given as the first statement of Theorem 1.66. Indeed, if $A = \mathbb{C}$ then the only automorphism of A is the trivial one, so that $\mathcal{N}(F)(s,t) = F(ts^{-1}) = N(F)(s,t)$ $(s, t \in G)$. Thus, in this case, the statement of Theorem 3.11 reduces to classical transference by Remark 2.4 and Remark 3.2.

The following continuity property will be used in the next result.

Definition 3.13. Let $F : G \to \mathcal{B}(A)$ be a bounded, pointwise-measurable function and (ρ, τ) a covariant representation of (A, G, α) on the Hilbert space $\mathcal{H}_{\rho,\tau}$. We say that F is (ρ, τ)-*fibre-continuous* if, for every $a \in A$, the map

$$s \mapsto \rho\big(F(s)(a)\big)\tau_s$$

is weak*-continuous from G to $\mathcal{B}(\mathcal{H}_{\rho,\tau})$.

The following result explains the link between Herz–Schur (A, G, α)-multipliers and Herz–Schur θ-multipliers of (A, G, α).

Corollary 3.14. *Let (A, G, α) be a C^*-dynamical system, $(\theta, \mathcal{H}_\theta)$ a faithful representation of A on a separable Hilbert space, and $F : G \to \mathcal{CB}(A)$ a bounded, pointwise-measurable function. The following are equivalent:*

i. F is a Herz–Schur (A, G, α)-multiplier such that S_F^θ can be extended to a weak-continuous map on $A \rtimes_{\alpha,\theta}^{\mathrm{w}^*} G$;*

ii. there exists a completely bounded, weak-continuous map Φ on $A \rtimes_{\alpha,\theta}^{\mathrm{w}^*} G$ such that for almost all $t \in G$*

$$\Phi\big(\pi^\theta(a)\lambda_t^\theta\big) = \pi^\theta\big(F(t)(a)\big)\lambda_t^\theta, \quad a \in A.$$

In particular, the conditions hold if $\mathcal{N}(F)$ is a Schur θ-multiplier of A.

If, in addition, F is $(\pi^\theta, \lambda^\theta)$-fibre-continuous then condition (ii) is equivalent to F being a Herz–Schur θ-multiplier of (A, G, α). Under this additional assumption F is a Herz–Schur θ-multiplier of (A, G, α) if and only if F is a Herz–Schur multiplier of (A, G, α) such that S_F^θ has a weak-continuous extension to $A \rtimes_{\alpha,\theta}^{\mathrm{w}^*} G$.*

Proof. (i)⇒(ii) Suppose that $F : G \to \mathcal{CB}(A)$ is a Herz–Schur (A, G, α)-multiplier with Φ the weak*-continuous extension of S_F^θ to $A \rtimes_{\alpha,\theta}^{\mathrm{w}^*} G$. By Lemma 3.10 there exists a null set $M \subseteq G$ such that, if $(f_U)_U$ is a Dirac family at $t \in G \setminus M$ then, for any $a \in A$,

$$\Phi\big(\pi^\theta \rtimes \lambda^\theta(f_U \otimes a)\big) \xrightarrow{U} \pi^\theta\big(F(t)(a)\big)\lambda_t^\theta$$

in the weak operator topology. On the other hand, by Lemma 3.10,

$$\pi^\theta \rtimes \lambda^\theta(f_U \otimes a) \xrightarrow{U} \pi^\theta(a)\lambda_t^\theta$$

in the weak operator topology, so that for any $\xi, \eta \in L^2(G, \mathcal{H}_\theta)$ and $t \in G \setminus M$

$$\Big\langle \Phi\big(\pi^\theta(a)\lambda_t^\theta\big), \eta \Big\rangle = \Big\langle \pi^\theta\big(F(t)(a)\big)\lambda_t^\theta, \eta \Big\rangle.$$

It follows that (ii) holds for all $t \in G \setminus M$.

(ii)$\Rightarrow$(i) By Remark 3.4, if $f \in L^1(G, A)$ then, for any $\omega \in \mathcal{B}(L^2(G, \mathcal{H}_\theta))_*$,

$$\left\langle \Phi\left(\int_G \pi^\theta\big(f(t)\big)\lambda_t^\theta\, dt\right), \omega\right\rangle = \int_G \left\langle \pi^\theta\Big(F(t)\big(f(t)\big)\Big)\lambda_t^\theta, \omega\right\rangle\, dt.$$

Since F is assumed to be pointwise-measurable we have $F \cdot f \in L^1(G, A)$, so in this case it follows that

$$\begin{aligned}\left\langle \pi^\theta \rtimes \lambda^\theta(F \cdot f), \omega\right\rangle &= \left\langle \int_G \pi^\theta\Big(F(s)\big(f(s)\big)\Big)\lambda_s^\theta\, ds, \omega\right\rangle \\ &= \int_G \left\langle \pi^\theta\Big(F(s)\big(f(s)\big)\Big)\lambda_s^\theta, \omega\right\rangle\, ds \\ &= \left\langle \Phi\big(\pi^\theta \rtimes \lambda^\theta(f)\big), \omega\right\rangle,\end{aligned}$$

and therefore

$$\Phi\big(\pi^\theta \rtimes \lambda^\theta(f)\big) = \pi^\theta \rtimes \lambda^\theta(F \cdot f);$$

that is, Φ is a weak*-continuous extension of S_F^θ to $A \rtimes_{\alpha,\theta}^{\mathrm{w}^*} G$. Since Φ is completely bounded F is a Herz–Schur (A, G, α)-multiplier, as this property is independent of the faithful representation θ by Remark 3.2.

Now suppose that $\mathcal{N}(F)$ is a Schur θ-multiplier of A, so the map $S_{\mathcal{N}(F)^\theta}$ has a weak*-continuous extension to a completely bounded map on $\mathcal{B}(L^2(G)) \mathbin{\overline{\otimes}} \theta(A)''$. As in the proof of Theorem 3.11 let $(K_n)_{n\in\mathbb{N}}$ be an increasing sequence of compact sets with $G = \cup_{n\in\mathbb{N}} K_n$, $h_K \in L^2(G \times G, \theta(A))$ be given by (3.15), and $f \in C_c(G, A)$; then $T_{h_{K_n}} \to \pi^\theta \rtimes \lambda^\theta(f)$ in the weak* topology. As $S_{\mathcal{N}(F)^\theta}$ has a weak*-continuous extension to $\mathcal{B}(L^2(G)) \mathbin{\overline{\otimes}} \theta(A)''$ we have

$$\left\langle S_{\mathcal{N}(F)^\theta}(T_{h_{K_n}})\xi, \eta\right\rangle \to \left\langle S_{\mathcal{N}(F)^\theta}\big(\pi^\theta \rtimes \lambda^\theta(f)\big)\xi, \eta\right\rangle, \quad \xi, \eta \in L^2(G, \mathcal{H}_\theta).$$

Also, by (3.18),

$$\left\langle S_{\mathcal{N}(F)^\theta}(T_{h_{K_n}})\xi, \eta\right\rangle \to \left\langle S_F^\theta\big(\pi^\theta \rtimes \lambda^\theta(f)\big)\xi, \eta\right\rangle.$$

Thus S_F^θ is the restriction of $S_{\mathcal{N}(F)^\theta}$ to $A \rtimes_{\alpha,\theta} G$, so S_F^θ does have a weak*-continuous extension to $A \rtimes_{\alpha,\theta}^{\mathrm{w}^*} G$.

Finally, if F is a Herz–Schur θ-multiplier of (A, G, α) then the map Φ_F^θ of Definition 3.3 clearly satisfies (ii). Now suppose F is $(\pi^\theta, \lambda^\theta)$-fibre-continuous and (ii) holds. We must show that the almost everywhere condition of (ii) implies everywhere equality as required by Definition 3.3. If a sequence $(s_n)_{n\in\mathbb{N}} \subseteq G$ converges to $t \in G$ then clearly, for each $a \in A$, $(\pi^\theta(a)\lambda^\theta_{s_n})_{n\in\mathbb{N}}$ converges to $\pi^\theta(a)\lambda^\theta_t$ in the weak* topology; also, by $(\pi^\theta, \lambda^\theta)$-fibre-continuity, $(\pi^\theta(F(s_n)(a))\lambda^\theta_{s_n})_{n\in\mathbb{N}}$ converges to $\pi^\theta(F(t)(a))\lambda^\theta_t$ in the weak* topology. Since Φ is weak*-continuous it follows that

$$\Phi\big(\pi^\theta(a)\lambda^\theta_t\big) = \pi^\theta\big(F(t)(a)\big)\lambda^\theta_t, \quad a \in A,\ t \in G.$$

Thus the equivalence of (i) and (ii) becomes the claimed result. □

Corollary 3.15. *Let (A, G, α) be a C^*-dynamical system, $(\theta, \mathcal{H}_\theta)$ a faithful representation of A on a separable Hilbert space, and $F : G \to \mathcal{CB}(A)$ a Herz–Schur θ-multiplier of (A, G, α). Then $\sup_{t\in G} \|F(t)\|_{\mathrm{cb}} \le \|F\|_{\mathrm{HS}}$.*

Proof. If F is a Herz–Schur θ-multiplier of (A, G, α) then the map Φ_F^θ of Definition 3.3 satisfies (ii) of Corollary 3.14. For each $n \in \mathbb{N}$ we have

$$\begin{aligned}
\|\Phi^{(n)}\| &\ge \sup_{t\in G}\sup_{(a_{i,j})\neq 0} \frac{\left\|\Phi^{(n)}\Big(\big(\pi^\theta(a_{i,j})\lambda^\theta_t\big)_{i,j=1}^n\Big)\right\|}{\left\|\big(\pi^\theta(a_{i,j})\lambda^\theta_t\big)_{i,j=1}^n\right\|} \\
&= \sup_{t\in G}\sup_{(a_{i,j})\neq 0} \frac{\|\big(F(t)(a_{i,j})\big)_{i,j=1}^n\|}{\|(a_{i,j})_{i,j=1}^n\|} \\
&= \sup_{t\in G} \|F(t)^{(n)}\|,
\end{aligned}$$

since π^θ and λ^θ_t are isometries. The claim follows. □

We now turn to the task of identifying the image of the map $\mathcal{N}$. Recall from Subsection 1.4.2 that ρ^G, the right regular representation of G on $L^2(G)$, is given by

$$\rho^G : G \to \mathcal{B}(L^2(G));\ (\rho^G_t\xi)(s) := \Delta(t)^{\frac{1}{2}}\xi(st), \quad s, t \in G,\ \xi \in L^2(G),$$

and recall that $\operatorname{Ad} U$ denotes conjugation by U, *i.e.* $\operatorname{Ad} U(T) := UTU^*$. For each $t \in G$ define

$$\tilde{\alpha}_t : \mathcal{K}(L^2(G)) \otimes_{\min} A \to \mathcal{K}(L^2(G)) \otimes_{\min} A;\ \tilde{\alpha}_t := \operatorname{Ad} \rho_t^G \otimes \alpha_t.$$

By Kadison–Ringrose [35, Theorem 11.1.3] $\tilde{\alpha}_t$ is an automorphism of $\mathcal{K}(L^2(G)) \otimes_{\min} A$ for each $t \in G$.

Definition 3.16. A Schur A-multiplier $\varphi \in \mathfrak{S}_0(G, G; A)$ will be called *invariant* if S_φ commutes with $\tilde{\alpha}_r$ for every $r \in G$. The collection of all invariant Schur A-multipliers in $\mathfrak{S}_0(G, G; A)$ will be denoted $\mathfrak{S}_{\mathrm{inv}}(G, G; A)$.

It is clear from the definition of $N(F)$ that $N(F)(s,t) = N(F)(sr, tr)$ $(r, s, t \in G)$. If w is any function defined on $G \times G$, and $r \in G$, we define

$$w_r(s,t) := w(sr, tr).$$

The next two Lemmas investigate properties of maps of this form, and the interaction of T_k with $\tilde{\alpha}$.

Lemma 3.17. *Let $k \in L^2(G \times G, A)$ and $r \in G$. Then $\tilde{\alpha}_r(T_k) = T_{\tilde{k}}$, where $\tilde{k} \in L^2(G \times G, A)$ is given by*

$$\tilde{k}(s,t) = \Delta(r)\alpha_r\big(k_r(t,s)\big), \quad s,t \in G.$$

Proof. Let $k \in L^2(G \times G, A)$. Then

$$\|\tilde{k}\|_2^2 = \int_{G\times G} \Delta(r)^2 \|\alpha_r\big(k(tr,sr)\big)\|^2 \, d(t,s) = \int_{G\times G} \|k(t',s')\|^2 \, d(t',s'),$$

so $\tilde{k} \in L^2(G \times G, A)$ and $\|\tilde{k}\|_2 = \|k\|_2$.

Define two maps $\Theta, \Theta_r : L^2(G \times G, A) \to \mathcal{B}(L^2(G, \mathcal{H}))$ by

$$\Theta(k) := T_{\tilde{k}}, \quad \Theta_r(k) := \tilde{\alpha}_r(T_k).$$

By Lemma 2.1 and the first paragraph of this proof Θ and Θ_r are continuous. Now suppose that $k \in L^2(G\times G, A)$ is an elementary tensor $k = h \otimes a$, where $h \in L^2(G \times G)$

and $a \in A$. For $\xi \in L^2(G)$ we have

$$\begin{aligned}(\rho_r^G T_h \rho_{r^{-1}}^G \xi)(s) = \Delta(r)^{\frac{1}{2}} (T_h \rho_{r^{-1}}^G) \xi(sr) &= \Delta(r)^{\frac{1}{2}} \int_G h(sr, x)(\rho_{r^{-1}}^G \xi)(x)\, dx \\ &= \int_G h(sr, x) \xi(xr^{-1})\, dx \\ &= \int_G h(sr, yr) \xi(y) \Delta(r)\, dy;\end{aligned}$$

i.e. $\rho_r^G T_h \rho_{r^{-1}}^G = T_{\tilde{h}_r}$, where $\tilde{h}_r(s,t) = \Delta(r) h(sr, tr)$ $((s,t) \in G \times G)$. Thus

$$\tilde{\alpha}_r(T_k) = \tilde{\alpha}_r(T_h \otimes a) = T_{\tilde{h}_r} \otimes \alpha_r(a),$$

so $\Theta(k) = \Theta_r(k)$. The result follows by linearity and continuity since elementary tensors $k = h \otimes a$ span a dense subset of $L^2(G \times G, A)$ by Proposition 1.79. $\square$

The following Lemma improves a recent result of Todorov and Turowska [57, Lemma 3.9].

Lemma 3.18. *Let E be a separable Banach space and let $w : G \times G \to \mathcal{B}(E)$ be a bounded, pointwise-measurable function such that, for every $r \in G$, $w_r = w$ almost everywhere. Then there exists a bounded, pointwise-measurable function $u : G \to \mathcal{B}(E)$ such that $w = N(u)$ up to a null set.*

Proof. The map

$$\phi : G \times G \to G \times G;\ \phi(s,t) = (s, ts), \quad (s,t) \in G \times G,$$

is continuous, bijective, and preserves null sets in both directions (see *e.g.* Cohn [13, Lemma 9.4.3]). We have that $w_r(\phi(s,x)) = w(\phi(s,x))$ for all $r \in G$ and almost all $(s,x) \in G \times G$; *i.e.* $w(sr, xsr) = w(s, xs)$ for almost all $(s,x) \in G \times G$. We will show $w(sr, xsr) = w(s, xs)$ for almost all $(s,x,r) \in G \times G \times G$. Let $E_0 \subseteq E$ be a countable dense subset; for every $a \in E_0$ we have

$$\begin{aligned}
&\int_{G\times G\times G} \|w(sr, xsr)(a) - w(s, xs)(a)\| \, d(x, s, r) \\
&\quad = \int_G \left(\int_{G\times G} \|w(sr, xsr)(a) - w(s, xs)(a)\| \, d(x, s) \right) dr \\
&\quad = 0.
\end{aligned}$$

Thus there is a null set $M_a \subseteq G \times G \times G$ such that $w(sr, xsr)(a) = w(s, xs)(a)$ for all $(s, x, r) \notin M_a$. Let $M = \cup_{a\in E_0} M_a$. Then $w(sr, xsr)(a) = w(s, xs)(a)$ for all $(s, x, r) \notin M$ and all $a \in E_0$; since $w(sr, xsr)$ and $w(s, xs)$ are bounded operators on E this implies that $w(sr, xsr) = w(s, xs)$ for all $(s, x, r) \notin M$. Therefore we can choose $s_0 \in G$ such that

$$w(s_0 r, xs_0 r) = w(s_0, xs_0) \tag{3.19}$$

for almost all $(x, r) \in G \times G$. Define

$$u : G \to \mathcal{B}(E);\ u(t) := w(s_0, ts_0), \quad t \in G,$$

which is a bounded function; it follows from the pointwise-measurability of w that $t \mapsto u(t)(a)$ is measurable for all $a \in A$. We have $u(t) = w(y, ty)$ for almost all $(y, t) \in G \times G$ by (3.19). Letting $\tilde{u} : G \times G \to \mathcal{B}(E)$ be the function $\tilde{u}(s, t) := u(t)$ ($(s, t) \in G \times G$) we have that $w(y, xy) = \tilde{u}(y, x)$ for almost all $(x, y) \in G \times G$. It follows that $w(\phi^{-1}(y, xy)) = \tilde{u}(\phi^{-1}(y, x))$ for almost all $(x, y) \in G \times G$; that is, $w(y, x) = u(xy^{-1})$ for almost all $(x, y) \in G \times G$. □

The following Lemma is similar to Proposition 1.65.

Lemma 3.19. *Let $\varphi \in \mathfrak{S}_0(G, G; A)$. The following are equivalent:*

i. φ is an invariant Schur A-multiplier;

ii. for every $r \in G$, $\mathcal{T}(\varphi)_r = \mathcal{T}(\varphi)$ almost everywhere.

Proof. In this proof we assume that A is faithfully represented on a separable Hilbert space.

(i)⇒(ii) Let $r \in G$ and assume $k \in L^2(G \times G, A)$ is of the form $k = h \otimes a$ for some $h \in L^2(G\times G)$ and $a \in A$. By Lemma 3.17 $(S_\varphi \circ \tilde{\alpha}_r)(T_h \otimes a) = T_{k_1}$, where $k_1 : G\times G \to A$ is given by

$$k_1(s,t) := \Delta(r)h(sr,tr)\varphi(t,s)\big(\alpha_r(a)\big), \quad s,t \in G,$$

while $(\tilde{\alpha}_r \circ S_\varphi)(T_h \otimes a) = T_{k_2}$, where $k_2 : G \times G \to A$ is given by

$$k_2(s,t) := \Delta(r)h(sr,tr)\alpha_r\big(\varphi(tr,sr)(a)\big), \quad s,t \in G.$$

By Lemma 2.1 and our assumption (i), $k_1 = k_2$ almost everywhere. Hence

$$\varphi(sr,tr)(a) = \alpha_{r^{-1}}\Big(\varphi(s,t)\big(\alpha_r(a)\big)\Big),$$

for almost all $(s,t) \in G \times G$. Thus, for every $a \in A$,

$$\begin{aligned} \mathcal{T}(\varphi)(sr,tr)(a) &= \alpha_{tr}\Big(\varphi(sr,tr)\big(\alpha_{r^{-1}t^{-1}}(a)\big)\Big) \\ &= \alpha_{tr}\Big(\alpha_{r^{-1}}\Big(\varphi(s,t)\big(\alpha_r\big(\alpha_{r^{-1}t^{-1}}(a)\big)\big)\Big)\Big) \\ &= \alpha_t\Big(\varphi(s,t)\big(\alpha_{t^{-1}}(a)\big)\Big) \end{aligned}$$

for almost all $(s,t) \in G \times G$. Since A is separable we conclude that $\mathcal{T}(\varphi)_r(s,t) = \mathcal{T}(\varphi)(s,t)$ for almost all $(s,t) \in G \times G$.

(ii)⇒(i) Using the assumption (ii) and reversing the above steps we obtain $k_1 = k_2$ almost everywhere. Since operators of the form $T_h \otimes a$ have dense linear span in $\mathcal{K}(L^2(G)) \otimes_{\min} A$ we obtain that S_φ commutes with $\tilde{\alpha}_r$, *i.e.* φ is an invariant Schur A-multiplier. □

Now we are able to identify the image of $\mathcal{N}$, similarly to the second part of Theorem 1.66.

Theorem 3.20. *Let (A, G, α) be a C^*-dynamical system. The map $\mathcal{N}$ is a linear isometry from $\mathfrak{S}(A, G, \alpha)$ onto $\mathfrak{S}_{\mathrm{inv}}(G, G; A)$.*

Proof. By Theorem 3.11 the map $\mathcal{N}$ is a linear isometry of $\mathfrak{S}(A,G,\alpha)$ into $\mathfrak{S}_0(G,G;A)$. By definition $\mathcal{N}=\mathcal{T}^{-1}\circ N$ so we have, for almost all $(s,t)\in G\times G$,

$$\mathcal{T}\big(\mathcal{N}(F)\big)_r(s,t)=\mathcal{T}\big(\mathcal{N}(F)\big)(sr,tr)=N(F)(sr,tr)=F(ts^{-1})=\mathcal{T}\big(\mathcal{N}(F)\big)(s,t),$$

so $\mathcal{T}(\mathcal{N}(F))_r=\mathcal{T}(\mathcal{N}(F))$ almost everywhere, for every $r\in G$ and all $F\in\mathfrak{S}(A,G,\alpha)$. It follows from Lemma 3.19 that the image of $\mathcal{N}$ is contained in $\mathfrak{S}_{\mathrm{inv}}(G,G;A)$.

To show $\mathcal{N}$ is surjective let $(\theta,\mathcal{H}_\theta)$ be a faithful representation of A on a separable Hilbert space. Take $\varphi\in\mathfrak{S}_{\mathrm{inv}}(G,G;A)$. By Lemma 3.19 and Lemma 3.18 there is a bounded, pointwise-measurable function $F:G\to\mathcal{B}(A)$ such that $N(F)=\mathcal{T}(\varphi)$ almost everywhere. It follows that $\mathcal{N}(F)=\varphi$ almost everywhere. Since $\varphi(s,t)$ is completely bounded for all $(s,t)\in G\times G$ we have that $F(t)\in\mathcal{CB}(A)$ for all $t\in G$. Since φ is a Schur A-multiplier the proof of Theorem 3.11 shows that the map

$$\pi^\theta\rtimes\lambda^\theta(f)\mapsto\pi^\theta\rtimes\lambda^\theta(F\cdot f),\quad f\in L^1(G,A),$$

is completely bounded. Thus F is a Herz–Schur (A,G,α)-multiplier. □

3.3 Connections with other multipliers

We now investigate how our notion of multipliers of a C^*-dynamical system is related to two other notions which have appeared in the literature. The first of these definitions turns out to be a special case of the Herz–Schur multipliers defined in Chapter 3, while the second definition, given only for discrete groups, is equivalent to our definition of Herz–Schur multipliers of a C^*-dynamical system.

3.3.1 Multipliers of Dong and Ruan

Dong and Ruan [20, Section 3] have introduced multipliers of a discrete C^*-dynamical system in order to study a version of the Haagerup property for C^*-dynamical systems. Let us first describe briefly the notation and assumptions in their paper. Suppose that (A, G, α) is a C^*-dynamical system with G a discrete group. It is well known (*e.g.* Brown–Ozawa [11, page 118]) that in this case one may define an operator on $A \rtimes_{\alpha,r} G$ by defining the action on finite sums $\sum \pi(a_t)\lambda_t$ $(a_t \in A,\ t \in G)$, which form a dense subspace of $A \rtimes_{\alpha,r} G$ when G is discrete. Moreover it is common to omit the representation π when writing such sums. Dong–Ruan use these technical and notational simplifications to write elements from a dense subspace of $A \rtimes_{\alpha,r} G$ as finite sums $\sum_t \lambda_t a_t$ $(a_t \in A,\ t \in G)$; the order of $a \in A$ and λ_t is only a matter of convention. These comments show that the following definition is identical to that given by Dong–Ruan [20, Section 3].

Definition 3.21. Let (A, G, α) be a C^*-dynamical system with G a discrete group. A function $h : G \to A$ is called a *multiplier of* (A, G, α) *in the sense of Dong and Ruan*, or a *DR-multiplier of* (A, G, α), if there exists a bounded $\pi(A)$-bimodule map Φ_h on $A \rtimes_{\alpha,r} G$ satisfying $\Phi_h(\lambda_s) = \pi(h(s))\lambda_s$ for all $s \in G$. If the map Φ_h is completely bounded then we call h a *completely bounded DR-multiplier of* (A, G, α).

If h is a DR-multiplier of (A, G, α) then h necessarily takes values in the centre of A.

Suppose that h is a DR-multiplier of (A, G, α) and define

$$F_h : G \to \mathcal{CB}(A);\ F_h(t)(a) := h(t)a, \quad t \in G,\ a \in A.$$

Then, for any $f \in C_c(G, A)$, we have

$$\begin{aligned} \pi \rtimes \lambda(F_h \cdot f) &= \sum_{s \in G} \pi\big(F_h \cdot f(s)\big)\lambda_s \\ &= \sum_{s \in G} \pi\big(f(s)h(s)\big)\lambda_s \\ &= \sum_{s \in G} \Phi_h\Big(\pi\big(f(s)\big)\lambda_s\Big) \\ &= \Phi_h\big(\pi \rtimes \lambda(f)\big) \end{aligned}$$

so that F_h is a multiplier of (A, G, α). The same calculation shows that if h is a completely bounded DR-multiplier of (A, G, α) then F_h is a Herz–Schur (A, G, α)-multiplier.

We have shown what should be intuitively clear from comparing the definitions: multipliers in the sense of Dong and Ruan are a natural special case of Herz–Schur multipliers of a C^*-dynamical system, namely the case arising from the inclusion $A \subseteq \mathcal{CB}(A)$.

3.3.2 Multipliers of Bédos and Conti

Bédos and Conti [4, Section 4] have considered multipliers of a discrete (twisted) C^*-dynamical system. I will describe their main definitions and observe a link with the multipliers considered in this chapter. Let (A, G, α) be a C^*-dynamical system; although Bédos–Conti consider *twisted* C^*-dynamical systems I will assume the twist is trivial, to match the setting used so far in this thesis.

Let $T : G \times A \to A$ be a function which is linear in the second variable. For each $s \in G$ write

$$T_s : A \to A;\ T_s(a) := T(s, a), \quad a \in A.$$

For $f \in C_c(G, A)$ define

$$(T \cdot f)(s) := T_s\big(f(s)\big), \quad s \in G.$$

The following definition was given by Bédos–Conti [4, Section 4] in the case of discrete groups.

Definition 3.22. Let (A, G, α) be a C^*-dynamical system. A function $T : G \times A \to A$, which is linear in the second variable, will be called a *multiplier of* (A, G, α) *in the sense of Bédos and Conti*, or a *BC-multiplier of* (A, G, α), if there exists a bounded linear map

$$M_T : A \rtimes_{\alpha,r} G \to A \rtimes_{\alpha,r} G;\ M_T\big(\pi \rtimes \lambda(f)\big) := \pi \rtimes \lambda(T \cdot f), \quad f \in C_c(G, A). \tag{3.20}$$

In this case we define $\|T\|_{\mathrm{BC}} := \|M_T\|$. If the map M_T is completely bounded then we call T a *completely bounded BC-multiplier of* (A, G, α) and consider the norm $\|T\|_{\mathrm{BCcb}} := \|M_T\|_{\mathrm{cb}}$.

Observe the similarity of condition (3.20) and condition (3.1) used to define Herz–Schur multipliers of (A, G, α). Given a BC-multiplier of (A, G, α) $T : G \times A \to A$, define

$$F_T : G \to \mathcal{B}(A);\ F_T(t)(a) := T(t, a), \quad t \in G,\ a \in A.$$

Since T is a BC-multiplier of (A, G, α) we have $\|T(g, a)\| \leq \|M_T\|\|a\|$ for all $g \in G,\ a \in A$, so that F_T does indeed take values in $\mathcal{B}(A)$. Moreover, for $f \in C_c(G, A)$,

$$\begin{aligned} S_{F_T}\big(\pi \rtimes \lambda(f)\big) = \pi \rtimes \lambda(F_T \cdot f) &= \int_G \pi\Big(F_T(s)\big(f(s)\big)\Big)\lambda_s\, ds \\ &= \int_G \pi\Big(T_s\big(f(s)\big)\Big)\lambda_s\, ds \\ &= \pi \rtimes \lambda(T \cdot f) \\ &= M_T\big(\pi \rtimes \lambda(f)\big). \end{aligned}$$

Since $C_c(G, A)$ is dense in $L^1(G, A)$ it follows that if T is a BC-multiplier of (A, G, α) then F_T is a multiplier of (A, G, α) in the sense of Definition 3.1, and $\|T\|_{\mathrm{BC}} = \|F_T\|$. Reversing the above steps shows that if $F : G \to \mathcal{B}(A)$ is a multiplier of (A, G, α) then the map

$$T_F : G \times A \to A;\ T_F(t, a) := F(t)(a), \quad t \in G,\ a \in A,$$

is a BC-multiplier of (A, G, α). Repeating the above arguments for completely bounded multipliers shows that the completely bounded BC-multipliers of (A, G, α) are precisely the Herz–Schur (A, G, α)-multipliers, and that the norm is the same in this case.

Recall the definition of a Hilbert C^*-module from Subsection 1.1.3. The following definition is used by Bédos–Conti [4].

Definition 3.23. Let (A, G, α) be a C^*-dynamical system. An *equivariant representation* of (A, G, α) on a Hilbert A-module $\mathcal{M}$ is a pair (ρ, τ), where $\rho : A \to \mathcal{L}(\mathcal{M})$ is a representation of A on $\mathcal{M}$ (*i.e.* a homomorphism from A to the bounded linear operators on $\mathcal{M}$) and $\tau : G \to \mathcal{I}(\mathcal{M})$ is a group homomorphism from G into the group of invertible bounded linear operators on $\mathcal{M}$, satisfying the conditions:

i. $\rho\big(\alpha_s(a)\big) = \tau(s)\rho(a)\tau(s)^{-1}$ for all $s \in G,\ a \in A$;

ii. $\alpha_s\left(\langle \xi | \eta \rangle\right) = \langle \tau(s)\xi | \tau(s)\eta \rangle$ for all $s \in G,\ \xi, \eta \in \mathcal{M}$;

iii. $\tau(s)(\xi \cdot a) = \big(\tau(s)\xi\big) \cdot a$ for all $s \in G,\ \xi \in \mathcal{M},\ a \in A$;

iv. the map $s \mapsto \tau(s)\xi$ is continuous for every $\xi \in \mathcal{M}$.

The following result, stated in terms of completely bounded BC-multipliers, was proved by Bédos–Conti [4, Theorem 4.8] using a different method.

Corollary 3.24. *Let (ρ, τ) be an equivariant representation of the C^*-dynamical system (A, G, α) on a countably-generated Hilbert A-module $\mathcal{M}$, and let $\xi, \eta \in \mathcal{M}$. Define*

$$F : G \to \mathcal{CB}(A);\ F(t)(a) := \langle \xi | \rho(a)\tau(t)\eta \rangle, \quad t \in G,\ a \in A.$$

Then F is a Herz–Schur (A, G, α)-multiplier.

Proof. By Theorem 3.11 it suffices to show that $\mathcal{N}(F)$ is a Schur A-multiplier. For any $s, t \in G$ and $a \in A$ we have

$$\begin{aligned}
\mathcal{N}(F)(s,t)(a) &= \alpha_{t^{-1}}\Big(F(ts^{-1})\big(\alpha_t(a)\big)\Big) \\
&= \alpha_{t^{-1}}\Big(\,\big\langle \xi | \rho\big(\alpha_t(a)\big)\tau(ts^{-1})\eta \big\rangle\,\Big) \\
&= \big\langle \tau(t^{-1})\xi | \tau(t^{-1})\rho\big(\alpha_t(a)\big)\tau(ts^{-1})\eta) \big\rangle \\
&= \big\langle \tau(t^{-1})\xi | \rho(a)\tau(s^{-1})\eta \big\rangle\,.
\end{aligned}$$

For all $t \in G,\ \xi \in \mathcal{M}$ we have

$$\|\tau(t)\xi\|^2 = \|\,\langle \tau(t)\xi | \tau(t)\xi \rangle\,\| = \|\alpha_t\left(\langle \xi | \xi \rangle\right)\| = \|\,\langle \xi | \xi \rangle\,\| = \|\xi\|^2.$$

Since $t \mapsto \tau(t^{-1})\xi$ and $s \mapsto \tau(s^{-1})\eta$ are measurable the result follows from Theorem 2.11. ☐

Chapter 4

Several classes of multipliers

In this chapter we describe multipliers of certain 'types'. First we show how the classical multipliers of Section 1.4 automatically give rise to their vector-valued analogues developed in Chapters 2 and 3. The second section describes multipliers coming from elements of the Haagerup tensor product; we investigate such multipliers in our framework after briefly introducing the Haagerup tensor product of C^*-algebras. In the third section we are motivated by the applications of positive Herz–Schur multipliers in the literature, for example in the study of approximation properties of the reduced group C^*-algebra; see Lance [39, Proposition 4.1] and Haagerup [26, Lemma 1.1]. We aim to present a unified view of positivity for the vector-valued Schur and Herz–Schur multipliers considered in this thesis.

4.1 Classical multipliers

In this section we show how classical Schur and Herz–Schur multipliers can be viewed as vector-valued multipliers.

Proposition 4.1. *Let (X,μ) and (Y,ν) be standard measure spaces and $A \subseteq \mathcal{B}(\mathcal{H})$ a separable, non-degenerate C^*-algebra. Suppose $\varphi \in L^\infty(X \times Y)$ and define*

$$\varphi' : X \times Y \to \mathcal{CB}(A);\ \varphi'(x,y)(a) := \varphi(x,y)a, \quad (x,y) \in X \times Y,\ a \in A.$$

The following are equivalent:

i. φ' *is a Schur A-multiplier;*

ii. φ *is a Schur multiplier.*

Moreover, if the conditions hold then $\|\varphi'\|_{\mathfrak{S}} = \|\varphi\|_{\mathfrak{S}}$.

Proof. (i)⇒(ii) By Theorem 2.9 we have, for almost all $(x, y) \in X \times Y$ and any $a \in A$,

$$\varphi(x, y)a = \varphi'(x, y)(a) = W(y)^*\rho(a)V(x)$$

for some representation $(\rho, \mathcal{H}_\rho)$ of A on a separable Hilbert space and operators $V \in L^\infty(X, \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho))$, $W \in L^\infty(Y, \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho))$. Let $\xi \in \mathcal{H}$ be a unit vector and $(a_i)_{i\in\mathbb{N}}$ a bounded approximate identity for A. Then we have

$$\langle \varphi(x, y)a_i\xi, \xi\rangle = \langle \rho(a_i)V(x)\xi, W(y)\xi\rangle$$

for almost all $(x, y) \in X \times Y$. Since $A \subseteq \mathcal{B}(\mathcal{H})$ is non-degenerate and ρ is a non-degenerate representation, passing to a limit along i we obtain

$$\varphi(x, y) = \langle V(x)\xi, W(y)\xi\rangle, \quad \text{almost all } (x, y) \in X \times Y.$$

It follows from Theorem 1.58 that φ is a Schur multiplier.
(ii)⇒(i) Since $L^2(X)$ and $L^2(Y)$ are second-countable there exist, by Theorem 1.58, measurable functions $\xi : X \to \ell^2$ and $\eta : Y \to \ell^2$ such that

$$\varphi(x, y) = \langle \xi(x), \eta(y)\rangle$$

for almost all $(x, y) \in X \times Y$, and $\|\varphi\|_{\mathfrak{S}} = \text{ess sup}_{x\in X} \|\xi(x)\| \, \text{ess sup}_{y\in Y} \|\eta(y)\|$. Let $\mathcal{H}^\infty := \oplus_{i\in\mathbb{N}}\mathcal{H}$ and $\rho : A \to \mathcal{B}(\mathcal{H}^\infty)$ be the countable amplification of the identity representation of A; that is, $\rho(a) := a^\infty$, given by (3.6). Write $\xi(x) = (\xi_i(x))_{i\in\mathbb{N}}$ $(x \in X)$ and $\eta(y) = (\eta_i(y))_{i\in\mathbb{N}}$ $(y \in Y)$. Define

$$V(x) : \mathcal{H} \to \mathcal{H}^\infty; \; V(x) := \big(\xi_i(x)I_{\mathcal{H}}\big)_{i\in\mathbb{N}}, \quad x \in X, \tag{4.1}$$

and similarly

$$W(y) : \mathcal{H} \to \mathcal{H}^\infty;\ W(y) := \big(\eta_i(y) I_{\mathcal{H}}\big)_{i\in\mathbb{N}}, \quad y \in Y. \tag{4.2}$$

Clearly V and W are measurable; moreover, for any $x \in X$,

$$\|V(x)\|^2 = \|V(x)^*V(x)\| = \sum_{i=1}^{\infty} \overline{\xi_i(x)}\xi_i(x) = \|\xi(x)\|_2^2,$$

and similarly $\|W(y)\| = \|\eta(y)\|_2$ for any $y \in Y$, so that

$$\sup_{x\in X} \|V(x)\| = \sup_{x\in X} \|\xi(x)\|_2 \text{ and } \sup_{y\in Y} \|W(y)\| = \sup_{y\in Y} \|\eta(y)\|_2.$$

This shows that V and W are essentially bounded and therefore $V \in L^\infty(X, \mathcal{B}(\mathcal{H}, \mathcal{H}^\infty))$ and $W \in L^\infty(Y, \mathcal{B}(\mathcal{H}, \mathcal{H}^\infty))$. We have that

$$W(y)^*\rho(a)V(x) = \sum_{i=1}^{\infty} \xi_i(x)\overline{\eta_i(y)}a = \varphi(x, y)a = \varphi'(x, y)(a), \quad a \in A,$$

for almost all $(x, y) \in X\times Y$. It follows from Theorem 2.9 that φ' is a Schur A-multiplier.

For the norm equality observe that, from the proof of (i)⇒(ii), we have

$$\|\varphi\|_{\mathfrak{S}} \le \operatorname*{ess\,sup}_{x\in X} \|V(x)\| \ \operatorname*{ess\,sup}_{y\in Y} \|W(y)\| = \|\varphi'\|_{\mathfrak{S}},$$

where V and W are the operators associated to the Schur A-multiplier φ' in Theorem 2.9, chosen to satisfy the equality. On the other hand, if V and W are the maps defined in (4.1) and (4.2), we have shown above that, for any $x \in X$, $\|V(x)\| = \|\xi(x)\|_2$, and for any $y \in Y$ $\|W(y)\| = \|\eta(y)\|_2$. Thus $\|\varphi'\|_{\mathfrak{S}} \le \|\varphi\|_{\mathfrak{S}}$. □

Using this result and the Transference Theorems we can prove a similar result for Herz–Schur multipliers.

Proposition 4.2. *Let (A, G, α) be a C^*-dynamical system and $u : G \to \mathbb{C}$ a continuous, bounded function. Define*

$$F_u : G \to \mathcal{CB}(A);\ F_u(t)(a) := u(t)a, \quad t \in G,\ a \in A.$$

The following are equivalent:

i. F_u *is a Herz–Schur* (A, G, α)*-multiplier;*

ii. u *is a Herz–Schur multiplier.*

Moreover, if the conditions hold then $\|F_u\|_{\mathrm{HS}} = \|u\|_{\mathrm{M^{cb}}}$. *Finally, if the conditions hold then* F_u *is a Herz–Schur* θ*-multiplier of* (A, G, α) *for every faithful representation* $(\theta, \mathcal{H}_\theta)$ *of* A *on a separable Hilbert space.*

Proof. We have

$$\mathcal{N}(F_u)(s,t)(a) = u(ts^{-1})a, \quad a \in A,\ s,t \in G.$$

The main claim follows from Theorem 3.11, Proposition 4.1, and Theorem 1.66. The norm equality follows because each of these results preserves the norm.

Now suppose that the conditions hold, and let Ψ_u denote the weak*-continuous, completely bounded map on $\mathcal{B}(L^2(G))$ corresponding to u *via* classical transference (see Theorem 1.66 and Theorem 1.57). Let $(\theta, \mathcal{H}_\theta)$ be a faithful representation of A on a separable Hilbert space. We have that $\mathcal{N}(F_u)$ is a Schur θ-multiplier of A; indeed,

$$\mathcal{N}(F_u)^\theta(s,t)\big(\theta(a)\big) = u(ts^{-1})\theta(a), \quad a \in A,\ (s,t) \in G \times G,$$

and hence $S_{\mathcal{N}(F_u)^\theta} = \Psi_u|_{\mathcal{K}(L^2(G))} \otimes \mathrm{id}_{\theta(A)}$. It follows that $S_{\mathcal{N}(F_u)^\theta}$ is the restriction of the weak*-continuous map $\Psi_u \otimes \mathrm{id}_{\theta(A)''}$ to $\mathcal{K}(L^2(G)) \otimes_{\min} \theta(A)$, and $\mathcal{N}(F_u)$ is a Schur θ-multiplier of A as claimed. By Corollary 3.14 F_u is a Herz–Schur θ-multiplier of (A, G, α), since F_u is automatically $(\pi^\theta, \lambda^\theta)$-fibre-continuous. $\square$

4.2 Multipliers from the Haagerup tensor product

In this section we show how Schur and Herz–Schur multipliers arise from the Haagerup tensor product of a C^*-algebra with itself. We begin by briefly describing the features of the Haagerup tensor product required here. Throughout this section A denotes a separable, non-degenerate, C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space.

The *column operator space* over A, denoted by $C_\infty(A)$, is the norm-closure of the space of sequences $(a_i)_{i\in\mathbb{N}} \subseteq A$ with only finitely many non-zero entries, considered as a single column of a matrix indexed by $\mathbb{N}\times\mathbb{N}$; that is, those sequences $(a_i)_{i\in\mathbb{N}} \subseteq A$ such that $\sum_{i=1}^\infty a_i^* a_i$ converges in norm. The operator space structure on $C_\infty(A)$ is given by the identification of $M_n(C_\infty(A))$ with $C_\infty(M_n(A))$ $(n \in \mathbb{N})$. Similarly, the *row operator space* over A, denoted by $R_\infty(A)$, is the norm-closure of the space of sequences $(a_i)_{i\in\mathbb{N}} \subseteq A$ with only finitely many non-zero entries, considered as a single row of a matrix indexed by $\mathbb{N}\times\mathbb{N}$; that is, those sequences $(a_i)_{i\in\mathbb{N}} \subseteq A$ such that $\sum_{i=1}^\infty a_i a_i^*$ converges in norm. The operator space structure on $R_\infty(A)$ is given by the identification of $M_n(R_\infty(A))$ with $R_\infty(M_n(A))$ $(n \in \mathbb{N})$. We refer to Blecher–le Merdy [9, page 13] for further discussion of these spaces.

Now we define the Haagerup tensor product of two C^*-algebras. This definition is taken from Blecher–le Merdy [9, Proposition 1.5.6].

Definition 4.3. Let B and C be C^*-algebras. The *Haagerup tensor product* of B and C, denoted $B \otimes_{\mathrm{h}} C$, consists of convergent sums $\sum_{i=1}^\infty b_i \otimes c_i$, where $\sum_{i=1}^\infty b_i b_i^*$ and $\sum_{i=1}^\infty c_i^* c_i$ converge in norm. That is, $(b_i)_{i\in\mathbb{N}} \in R_\infty(B)$ and $(c_i)_{i\in\mathbb{N}} \in C_\infty(C)$. For $x \in B \odot C$ the norm of x is given by

$$\|x\|_{\mathrm{h}} = \inf \left\| \sum_{i=1}^m b_i b_i^* \right\|^{\frac{1}{2}} \left\| \sum_{i=1}^m c_i^* c_i \right\|^{\frac{1}{2}},$$

where the infimum is taken over all possible ways to write $x = \sum_{i=1}^m b_i \odot c_i$ in $B \odot C$.

Note that we have only defined the Haagerup tensor product of C^*-algebras; for the general operator space version see Blecher–le Merdy [9, 1.5.4] or Effros–Ruan [22, Chapter 9]. We will be focusing on the Haagerup tensor product when both components are the C^*-algebra A.

Observe that $A \otimes_{\mathrm{h}} A$ embeds canonically into $\mathcal{CB}(A)$ by identifying $\sum_{i=1}^\infty b_i \otimes a_i \in A \otimes_{\mathrm{h}} A$ with the completely bounded map $a \mapsto \sum_{i=1}^\infty b_i a a_i$ on A [12, page 65]. Indeed, let S

denote the map on A given by $a \mapsto \sum_{i=1}^{\infty} b_i a a_i$; we have, for any $\xi, \eta \in \mathcal{H}$ and any $a \in A$,

$$
\begin{aligned}
|\langle S(a)\xi, \eta\rangle| = \left|\sum_{i=1}^{\infty} \langle a a_i \xi, b_i^* \eta\rangle\right| &\leq \left(\sum_{i=1}^{\infty} \|a a_i \xi\|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} \|b_i^* \eta\|^2\right)^{\frac{1}{2}} \\
&\leq \|a\| \left(\sum_{i=1}^{\infty} \langle a_i^* a_i \xi, \xi\rangle\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} \langle b_i b_i^* \eta, \eta\rangle\right)^{\frac{1}{2}} \\
&\leq \|a\| \left\|\sum_{i=1}^{\infty} a_i^* a_i\right\| \left\|\sum_{i=1}^{\infty} b_i b_i^*\right\| \|\xi\| \|\eta\|.
\end{aligned} \tag{4.3}
$$

In particular, it follows from (4.3) that

$$
\left\|\sum_{i=k}^{m} b_i a a_i\right\| \leq \|a\| \left\|\sum_{i=k}^{m} a_i^* a_i\right\| \left\|\sum_{i=k}^{m} b_i b_i^*\right\|. \tag{4.4}
$$

Since $\sum_{i=1}^{\infty} a_i^* a_i$ and $\sum_{i=1}^{\infty} b_i b_i^*$ converge their partial sums form a Cauchy sequence, so by (4.4) the partial sums of $\sum_{i=1}^{\infty} b_i a a_i$ also form a Cauchy sequence in A; thus $S : A \to A$ is a bounded operator and $\|S\| \leq \|\sum_{i=1}^{\infty} b_i \otimes a_i\|_{\mathrm{h}}$. Moreover, let $n \in \mathbb{N}$ and $(a_{p,q})_{p,q=1}^{n} \in M_n(A)$, then

$$
S^{(n)}\big((a_{p,q})_{p,q=1}^{n}\big) = \sum_{i=1}^{\infty} \mathrm{diag}_n(b_i)\big((a_{p,q})_{p,q=1}^{n}\big)\mathrm{diag}_n(a_i), \tag{4.5}
$$

where $\mathrm{diag}_n(x)$ denotes the $n \times n$ matrix with all diagonal entries equal to x and all other entries 0. Since $\|\mathrm{diag}_n(x)\| = \|x\|$ $(x \in A)$ it follows from calculation (4.3) that S is a completely bounded map on A.

Now let (X, μ) and (Y, ν) be standard measure spaces, as considered in Chapter 2 (recall that we assume the underlying topology to be locally compact). Let $\beta : X \to C_\infty(A)$ and $\gamma : Y \to C_\infty(A)$ be bounded, measurable functions. Write $\beta(x) = (\beta_i(x))_{i \in \mathbb{N}}$ $(x \in X)$ and $\gamma(y) = (\gamma_i(y))_{i \in \mathbb{N}}$ $(y \in Y)$. Observe that each β_i belongs to $L^\infty(X, A)$, since β_i is clearly measurable and bounded; similarly $\gamma_i \in L^\infty(Y, A)$ for all $i \in I$. The identification above allows us to define

$$
\varphi_{\beta,\gamma} : X \times Y \to \mathcal{CB}(A);\ \varphi_{\beta,\gamma}(x, y) := \sum_{i=1}^{\infty} \gamma_i(y)^* \otimes \beta_i(x), \quad (x, y) \in X \times Y. \tag{4.6}
$$

The measurability of β_i and γ_i imply that the partial sums of (4.6) define measurable functions; since the series converges in norm Williams [60, Lemma B.17] implies that $\varphi_{\beta,\gamma}$ is measurable. Thus $\varphi_{\beta,\gamma}$ is pointwise-measurable.

Recall from Definition 2.13 that φ is called a Schur id-multiplier of A if the map S_φ extends to a weak*-continuous map on $\mathcal{B}(L^2(X), L^2(Y)) \mathbin{\overline{\otimes}} A''$.

Proposition 4.4. *Let (X,μ) and (Y,ν) be standard measure spaces and $A \subseteq \mathcal{B}(\mathcal{H})$ a separable C^*-algebra. Let $\beta : X \to C_\infty(A)$ and $\gamma : Y \to C_\infty(A)$ be bounded, measurable functions. Then $\varphi_{\beta,\gamma}$ is a Schur* id*-multiplier of A. Moreover,*

$$S_{\varphi_{\beta,\gamma}}(T) = \sum_{i=1}^{\infty} \gamma_i^* T \beta_i, \quad T \in \mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} A, \tag{4.7}$$

where the series converges in norm.

Proof. Observe that

$$\left\| \sum_{i=1}^{\infty} \beta_i^* \beta_i \right\| = \operatorname*{ess\,sup}_{x\in X} \left\| \sum_{i=1}^{\infty} \beta_i(x)^* \beta_i(x) \right\| = \operatorname*{ess\,sup}_{x\in X} \|\beta(x)\|^2,$$

and similarly $\|\sum_{i=1}^\infty \gamma_i^*\gamma_i\| = \operatorname{ess\,sup}_{y\in Y} \|\gamma(y)\|^2$. We have, similarly to (4.3), for any $\xi \in L^2(X,\mathcal{H})$ and $\eta \in L^2(Y,\mathcal{H})$,

$$\begin{aligned}
\left| \left\langle \sum_{i=1}^{\infty} \gamma_i^* T \beta_i \xi, \eta \right\rangle \right| &= \left| \sum_{i=1}^{\infty} \langle T\beta_i\xi, \gamma_i\eta\rangle \right| \\
&\le \left(\sum_{i=1}^{\infty} \|T\beta_i\xi\|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} \|\gamma_i\eta\|^2 \right)^{\frac{1}{2}} \\
&\le \|T\| \left\| \sum_{i=1}^{\infty} \beta_i^*\beta_i \right\| \|\xi\| \left\| \sum_{i=1}^{\infty} \gamma_i^*\gamma_i \right\| \|\eta\|,
\end{aligned}$$

since

$$\sum_{i=1}^{\infty} \|\gamma_i\eta\|^2 = \sum_{i=1}^{\infty} \langle \gamma_i^*\gamma_i\eta, \eta\rangle \le \left\| \sum_{i=1}^{\infty} \gamma_i^*\gamma_i \right\| \|\eta\|^2,$$

and similarly $\sum_{i=1}^\infty \|T\beta_i\xi\|^2 \le \|T\| \|\sum_{i=1}^\infty \gamma_i^*\gamma_i\| \|\xi\|^2$. It now follows, as in (4.4), that the series in (4.7) converges in norm.

Let $k \in L^2(Y \times X, A)$, $\xi \in L^2(X, \mathcal{H})$, and $\eta \in L^2(Y, \mathcal{H})$. Then, by Fubini's Theorem, Theorem 1.75,

$$\begin{aligned}
\left\langle \sum_{i=1}^{\infty} \gamma_i^* T_k \beta_i \xi, \eta \right\rangle &= \int_Y \sum_{i=1}^{\infty} \langle T_k(\beta_i \xi)(y), (\gamma_i \eta)(y) \rangle \, d\nu(y) \\
&= \int_Y \sum_{i=1}^{\infty} \left\langle \int_X k(y,x)\Big(\beta_i(x)\big(\xi(x)\big)\Big) \, d\mu(x), \gamma_i(y)\big(\eta(y)\big) \right\rangle d\nu(y) \\
&= \int_{X \times Y} \sum_{i=1}^{\infty} \langle k(y,x)\beta_i(x)\xi(x), \gamma_i(y)\eta(y) \rangle \, d(\mu \times \nu)(x,y) \\
&= \int_{X \times Y} \big\langle \varphi_{\beta,\gamma}(x,y)\big(k(y,x)\big)\xi(x), \eta(y) \big\rangle \, d(\mu \times \nu)(x,y) \\
&= \big\langle T_{\varphi_{\beta,\gamma} \cdot k} \xi, \eta \big\rangle .
\end{aligned}$$

It follows that $S_{\varphi_{\beta,\gamma}}(T_k) = \sum_{i=1}^{\infty} \gamma_i^* T_k \beta_i$ for all $k \in L^2(Y \times X, A)$. Since the right side defines a completely bounded map as in (4.5) we conclude that $\varphi_{\beta,\gamma}$ is a Schur A-multiplier. Since $S_{\varphi_{\beta,\gamma}}$ is bounded and $\mathcal{S}_2(X, Y; A)$ is dense in $\mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} A$ identity (4.7) follows. Since the map on the right side of (4.7) is clearly weak*-extendable we conclude that $\varphi_{\beta,\gamma}$ is a Schur id-multiplier of A. □

Recall that if φ is a Schur A-multiplier the Transference Theorem, Theorem 3.20, characterises when the restriction of S_φ to $A \rtimes_{\alpha,r} G$ gives a Herz–Schur (A, G, α)-multiplier. We now give a characterisation of when multipliers of the form $\varphi_{\beta,\gamma}$ have this property.

Proposition 4.5. *Let (A, G, α) be a C^*-dynamical system, and $\beta : G \to R_\infty(A)$ and $\gamma : G \to C_\infty(A)$ bounded, measurable functions. The following are equivalent:*

i. there exists a Herz–Schur (A, G, α)-multiplier F such that S_F coincides with the restriction of $S_{\varphi_{\beta,\gamma}}$ to $A \rtimes_{\alpha,r} G$;

ii. for every $a \in A$ the function $\varphi_a : G \times G \to A$ given by

$$\varphi_a(s,t) := \sum_{i=1}^{\infty} \alpha_t\big(\gamma_i(t)\big)^* a \alpha_t\big(\beta_i(s)\big), \quad (s,t) \in G \times G,$$

has the following property: for every $r \in G$ $\varphi_a(sr, tr) = \varphi_a(s,t)$ for almost all $(s,t) \in G \times G$.

Moreover, if the conditions hold then the map S_F extends to a completely bounded, weak-continuous map on $A \rtimes^{\mathrm{w}^*}_{\alpha,r} G$.*

Proof. (i)$\Rightarrow$(ii) By Proposition 4.4 the map $S_{\varphi_{\beta,\gamma}}$ has a weak*-continuous extension to a completely bounded map on $\mathcal{B}(L^2(G)) \mathbin{\overline{\otimes}} A''$. Since S_F is the restriction of $S_{\varphi_{\beta,\gamma}}$ it possesses a weak*-continuous extension to a completely bounded map on $A \rtimes^{\mathrm{w}^*}_{\alpha,r} G$.

Let $a \in A$ and $s \in G$. Define $\beta_i^s \in L^\infty(G, A)$ by $\beta_i^s(t) := \beta_i(s^{-1}t)$ $(t \in G)$, so that for any $\xi \in L^2(G, \mathcal{H})$, $r \in G$,

$$\lambda_s \beta_i \xi(r) = \beta_i \xi(s^{-1}r) = \beta_i(s^{-1}r)\big(\xi(s^{-1}r)\big) = \beta_i^s \lambda_s \xi(r).$$

Thus $\lambda_s \beta_i = \beta_i^s \lambda_s$. By Corollary 3.14 we have, for almost all $s \in G$,

$$\pi\big(F(s)(a)\big) = \left(\sum_{i=1}^{\infty} \gamma_i^* \pi(a) \lambda_s \beta_i\right) (\lambda_s)^* = \sum_{i=1}^{\infty} \gamma_i^* \pi(a) \beta_i^s, \quad a \in A.$$

Therefore, if $\xi \in L^2(G, \mathcal{H})$ then, for almost all $s, t \in G$, we have

$$\begin{aligned}
\alpha_{t^{-1}}\big(F(s)(a)\big)\big(\xi(t)\big) &= \pi\big(F(s)(a)\big)\xi(t) \\
&= \left(\sum_{i=1}^{\infty} \gamma_i^* \pi(a) \beta_i^s \xi\right)(t) \\
&= \sum_{i=1}^{\infty} \gamma_i(t)^* \alpha_{t^{-1}}(a) \beta_i(s^{-1}t)\big(\xi(t)\big).
\end{aligned}$$

Since $\mathcal{H}$ is separable it has a countable orthonormal basis $(e_i)_{i\in\mathbb{N}}$. If G has finite Haar measure then the above calculation, with $\xi = \chi_G \otimes e_i \in L^2(G, \mathcal{H})$, implies that

$$\alpha_{t^{-1}}\big(F(s)(a)\big)e_i = \sum_{i=1}^{\infty} \gamma_i(t)^* \alpha_{t^{-1}}(a) \beta_i(s^{-1}t) e_i. \tag{4.8}$$

Since (4.8) holds for all $i \in \mathbb{N}$ we have

$$\alpha_{t^{-1}}\big(F(s)(a)\big) = \sum_{i=1}^{\infty} \gamma_i(t)^* \alpha_{t^{-1}}(a) \beta_i(s^{-1}t). \tag{4.9}$$

If the measure of G is not finite then choose, by σ-finiteness of G, a sequence $(K_n)_{n\in\mathbb{N}}$ of sets of finite measure such that $\cup_{n\in\mathbb{N}} K_n = G$. Given $t \in G$ find $m \in \mathbb{N}$ such that $t \in K_m$, and apply the above argument with $\xi = \chi_{K_m} \otimes e_i$. Now, since the series in (4.9) converges in norm, we have

$$\begin{aligned}
F(s)(a) &= \alpha_t\left(\sum_{i=1}^{\infty} \gamma_i(t)^* \alpha_{t^{-1}}(a)\beta_i(s^{-1}t)\right)\\
&= \alpha_t\left(\lim_{n\to\infty}\sum_{i=1}^{n} \gamma_i(t)^* \alpha_{t^{-1}}(a)\beta_i(s^{-1}t)\right)\\
&= \lim_{n\to\infty}\alpha_t\left(\sum_{i=1}^{n} \gamma_i(t)^* \alpha_{t^{-1}}(a)\beta_i(s^{-1}t)\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \alpha_t\big(\gamma_i(t)^*\big) a \alpha_t\big(\beta_i(s^{-1}t)\big)\\
&= \sum_{i=1}^{\infty} \alpha_t\big(\gamma_i(t)^*\big) a \alpha_t\big(\beta_i(s^{-1}t)\big);
\end{aligned}$$

that is, $\varphi_a(t, s^{-1}t) = F(s)(a)$ for almost all $s, t \in G$. Since the map $(s,t) \mapsto (t, s^{-1}t)$ is continuous, bijective, and preserves null sets in both directions [13, Lemma 9.4.3] we obtain $\varphi_a(s,t) = F(st^{-1})(a)$ for almost all $(s,t) \in G \times G$. Hence, for each $r \in G$, $\varphi_a(sr, tr) = \varphi_a(s,t)$ for almost all $(s,t) \in G \times G$.
(ii)$\Rightarrow$(i) As $\gamma(t), \beta(s) \in C_\infty(A)$ for all $s, t \in G$ we have that

$$\varphi_{\beta,\gamma}(s,t)(a) = \lim_{n\to\infty}\sum_{i=1}^{n} \gamma_i(t)^* a \beta_i(s)$$

in norm; in particular, $\varphi_{\beta,\gamma}(s,t)(a) \in A$ for all $a \in A$. Hence

$$\begin{aligned}
\alpha_t\Big(\varphi_{\beta,\gamma}(s,t)\big(\alpha_{t^{-1}}(a)\big)\Big) &= \alpha_t\left(\lim_{n\to\infty}\sum_{i=1}^{n} \gamma_i(t)^* \alpha_{t^{-1}}(a)\beta_i(s)\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \alpha_t\big(\gamma_i(t)^*\big) a \alpha_t\big(\beta_i(s)\big)\\
&= \sum_{i=1}^{\infty} \alpha_t\big(\gamma_i(t)\big)^* a \alpha_t\big(\beta_i(s)\big)\\
&= \varphi_a(s,t),
\end{aligned}$$

for all $s,t \in G,\ a \in A$. Thus $\mathcal{T}(\varphi_{\beta,\gamma})(s,t)(a) = \varphi_a(s,t)$ for all $(s,t) \in G \times G$. Fix $r \in G$ and let $E \subseteq A$ be a countable dense set. For every $a \in E$ we have

$$\begin{aligned}\mathcal{T}(\varphi_{\beta,\gamma})_r(s,t)(a) &= \mathcal{T}(\varphi_{\beta,\gamma})(sr,tr)(a)\\ &= \varphi_a(sr,tr)\\ &= \varphi_a(s,t)\\ &= \mathcal{T}(\varphi_{\beta,\gamma})(s,t)(a)\end{aligned}$$

for almost all $(s,t) \in G \times G$; *i.e.* there exists a set $X \subseteq G \times G$, with null complement, such that $\mathcal{T}(\varphi_{\beta,\gamma})_r(s,t)(a) = \mathcal{T}(\varphi_{\beta,\gamma})(s,t)(a)$ for all $(s,t) \in X$ and all $a \in E$. Fix $(s,t) \in X$. Since the maps $\mathcal{T}(\varphi_{\beta,\gamma})_r(s,t)$ and $\mathcal{T}(\varphi_{\beta,\gamma})(s,t)$ are bounded we have that $\mathcal{T}(\varphi_{\beta,\gamma})_r(s,t)(a) = \mathcal{T}(\varphi_{\beta,\gamma})(s,t)(a)$ for all $a \in A$. Thus $\mathcal{T}(\varphi_{\beta,\gamma})_r = \mathcal{T}(\varphi_{\beta,\gamma})$ almost everywhere, for all $r \in G$. By Proposition 4.4 and Lemma 3.19 $\varphi_{\beta,\gamma}$ is an invariant Schur A-multiplier, so by Theorem 3.20 there is a Herz–Schur (A,G,α)-multiplier F such that $\mathcal{N}(F) = \varphi_{\beta,\gamma}$ almost everywhere. □

4.3 Positive multipliers

In this section we study what it means for Schur A-multipliers and Herz–Schur (A,G,α)-multipliers to be 'positive'. One motivation for this investigation is Lance's work on nuclearity and amenability [39, Section 4], as well as that of Haagerup and his collaborators, see *e.g.* [26], where positive Herz–Schur multipliers are used to characterise approximation properties for reduced group C^*-algebras. Indeed, although Lance and Haagerup do not explicitly mention Herz–Schur multipliers, they make use of functions defined on a group which give rise to completely bounded maps on the associated reduced group C^*-algebra (see Theorem 1.61) which we have taken as the defining property of Herz–Schur multipliers; moreover, the functions σ_i used by Lance [39, Proposition 4.1] are positive-definite functions on the group which give rise to completely positive maps on the reduced group C^*-algebra. These ideas have motivated several authors, such as Bédos–Conti [5] and Dong–Ruan [20], to consider positivity for multipliers of crossed products; indeed, the multipliers studied in Section 3.3 have positive versions, introduced to study approximation properties for crossed products. Connections between the positive multipliers introduced here and those from the literature are explored in Subsection 4.3.3. Here we aim to give a unified approach to positivity for the multipliers appearing in this thesis, generalising the work summarised in Subsection 4.3.3, and making tools available for the study of approximation properties of reduced crossed products. This motivation suggests that our work on positivity should:

i. generalise the notion of positivity for classical Schur and Herz–Schur multipliers (see Haagerup [26] for the Herz–Schur case);

ii. interact well with the Transference Theorem, Theorem 3.11;

iii. respect the 'functional calculus' for Schur multipliers $\varphi \mapsto S_\varphi$, similarly for Herz–Schur multipliers.

Unfortunately point (iii) seems to cause some difficulty when the measure spaces involved are not discrete — it is not clear in what 'almost everywhere' sense positivity of the kernel $k \in L^2(X \times X, A)$ is equivalent to positivity of the associated Hilbert–Schmidt operator $T_k \in \mathcal{B}(L^2(X, \mathcal{H}))$. For this reason we first consider positivity for the measurable multipliers which have been discussed so far, achieving goals (i) and (ii), and (iii) in part, before specialising to the discrete case where we can give further results towards (iii).

4.3.1 Measurable positivity

We begin in the setting of Chapter 2: (X, μ) is a standard measure space, A is a separable C^*-algebra which we assume acts on a separable Hilbert space $\mathcal{H}$. The only simplification is to assume $X = Y$, so that we may use the order structure on the C^*-algebra $\mathcal{K}(L^2(X)) \otimes_{\min} A$.

Let us begin by proving the completely positive analogue of Theorem 2.9, which is similar to Stinespring's Theorem, Theorem 1.23.

Theorem 4.6. *Let $\varphi : X \times X \to \mathcal{CB}(A, \mathcal{B}(\mathcal{H}))$ be a bounded function. The following are equivalent:*

i. φ *is a Schur A-multiplier with* S_φ *completely positive;*

ii. there exists a non-degenerate representation $(\rho, \mathcal{H}_\rho)$ *of* A *on a separable Hilbert space, and an operator* $V \in L^\infty(X, \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho))$*, such that*

$$\varphi(x,y)(a) = V(y)^*\rho(a)V(x), \quad a \in A,$$

for almost all $(x,y) \in X \times Y$*.*

Moreover, if the conditions hold then V *may be chosen so that*

$$\|\varphi\|_{\mathfrak{S}} = \left(\operatorname*{ess\,sup}_{x \in X} \|V(x)\| \right)^2.$$

Proof. We follow the proof of Theorem 2.9, using Stinespring's Theorem, Theorem 1.23, instead of the Haagerup–Paulsen–Wittstock Theorem, Theorem 1.18.
(i)$\Rightarrow$(ii) Since S_φ is completely positive, by Stinespring's Theorem, Theorem 1.23, there exist a representation $(\theta, \mathcal{H}_\theta)$ of A, and an operator $V_0 \in \mathcal{B}(L^2(X) \otimes \mathcal{H}, \mathcal{H}_\theta)$, such that

$$S_\varphi(T) = V_0^* \theta(T) V_0, \quad T \in \mathcal{K}(L^2(X)) \otimes A.$$

Since $L^2(X) \otimes \mathcal{H}$ and A are separable we may assume $\mathcal{H}_\theta$ is separable. By Lemma 2.8 there exists a non-degenerate representation $(\rho, \mathcal{H}_\rho)$ of A on a separable Hilbert space, and a unitary operator $U : \mathcal{H}_\theta \to L^2(X) \otimes \mathcal{H}_\rho$, such that

$$U\theta(b \otimes a)U^* = b \otimes \rho(a), \quad b \in \mathcal{K}(L^2(X)),\ a \in A.$$

Let $V_1 = UV_0 \in \mathcal{B}(L^2(X, \mathcal{H}_\theta), L^2(X, \mathcal{H}_\rho))$. Then, for $b \in \mathcal{K}(L^2(X)),\ a \in A$, we have

$$\begin{aligned} S_\varphi(b \otimes a) &= V_0^* \theta(b \otimes a) V_0 \\ &= V_0^* U^* \big(b \otimes \rho(a)\big) U V_0 \\ &= V_1^* \big(b \otimes \rho(a)\big) V_1. \end{aligned} \tag{4.10}$$

Set

$$\mathcal{S} := \overline{\operatorname{span}}\{T V_1 L^2(X, \mathcal{H}) : T \in \mathcal{K}(L^2(X)) \otimes_{\min} \rho(A)\}.$$

Clearly $\mathcal{S}$ is invariant under $\mathcal{K}(L^2(X)) \otimes_{\min} \rho(A)$, so the projection onto $\mathcal{S}$ has the form $I_{L^2(X)} \otimes E$ for some projection $E \in \rho(A)'$. We have, as in (2.11),

$$V_1 = (I_{L^2(X)} \otimes E)V_1. \tag{4.11}$$

Let $\tilde{\rho} := \mathrm{id} \otimes \rho : \mathcal{K}(L^2(X)) \otimes_{\min} A \to \mathcal{K}(L^2(X)) \otimes_{\min} \mathcal{B}(\mathcal{H}_\rho)$; by (4.10) and (4.11) we have

$$S_\varphi(T) = V_1^* \tilde{\rho}(T)(I_{L^2(X)} \otimes E)V_1, \quad T \in \mathcal{K}(L^2(X)) \otimes_{\min} \rho(A). \tag{4.12}$$

It is clear that if $c, d \in L^\infty(X)$ then

$$\tilde{\rho}\big((M_d^* \otimes I_{\mathcal{H}})T(M_c \otimes I_{\mathcal{H}})\big) = (M_d^* \otimes I_{\mathcal{H}_\rho})\tilde{\rho}(T)(M_c \otimes I_{\mathcal{H}_\rho}). \tag{4.13}$$

Let $V = (I_{L^2(X)} \otimes E)V_1$. It follows from (4.12) that

$$S_\varphi(T) = V^* \tilde{\rho}(T)V, \quad T \in \mathcal{K}(L^2(X)) \otimes_{\min} A. \tag{4.14}$$

Now identities (4.13) and (4.14), and Lemma 2.7, imply that

$$V^*(M_d^* \otimes I_{\mathcal{H}_\rho})\tilde{\rho}(T)V = (M_d^* \otimes I_{\mathcal{H}})V^* \tilde{\rho}(T)V, \quad d \in \mathcal{D}_X,\ T \in \mathcal{K}(L^2(X)) \otimes_{\min} A. \tag{4.15}$$

Thus

$$\big\langle \tilde{\rho}(T)V\xi, (M_d \otimes I_{\mathcal{H}_\rho})V\eta \big\rangle = \langle \tilde{\rho}(T)V\xi, V(M_d \otimes I_{\mathcal{H}})\eta \rangle$$

for all $\xi, \eta \in L^2(X, \mathcal{H})$. We conclude that

$$(I_{L^2(X)} \otimes E)(M_d \otimes I_{\mathcal{H}_\rho})V = (I_{L^2(X)} \otimes E)V(M_d \otimes I_{\mathcal{H}})$$

so that $(M_d \otimes I_{\mathcal{H}_\rho})V = V(M_d \otimes I_{\mathcal{H}})$ for all $d \in \mathcal{D}_X$. It follows from Theorem 1.82 and Remark 1.83 that $V \in L^\infty(X, \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho))$.

Now let $k \in L^2(X \times X)$ and $a \in A$. For any $\xi, \eta \in L^2(X, \mathcal{H})$ we have

$$\langle S_\varphi(T_k \otimes a)\xi, \eta \rangle = \int_{X \times X} k(y,x) \, \langle \varphi(x,y)(a)\xi(x), \eta(y) \rangle \; d(\mu \times \mu)(x,y) \tag{4.16}$$

On the other hand, by (4.14),

$$\begin{aligned}
\langle S_\varphi(T_k \otimes a)\xi, \eta\rangle &= \big\langle V^*\big(T_k \otimes \rho(a)\big)V\xi, \eta\big\rangle \\
&= \big\langle \big(T_k \otimes \rho(a)\big)V\xi, V\eta\big\rangle \\
&= \int_{X\times X} k(y,x)\,\big\langle \rho(a)\big(V(x)\xi(x)\big), V(y)\eta(y)\big\rangle \; d(\mu\times\mu)(x,y) \\
&= \int_{X\times X} k(y,x)\,\big\langle V(y)^*\rho(a)\big(V(x)\xi(x)\big), \eta(y)\big\rangle \; d(\mu\times\mu)(x,y).
\end{aligned}$$

We now conclude, exactly as in the proof of Theorem 2.9, that

$$\varphi(x,y)(a) = V(y)^*\rho(a)V(x), \quad a \in A,$$

for almost all $(x,y) \in X \times Y$.

(ii)$\Rightarrow$(i) Condition (ii) implies that the map $S_\varphi : \mathcal{S}_2(X,X;A) \to \mathcal{S}_2(X,X;\mathcal{B}(\mathcal{H}))$ satisfies

$$S_\varphi(T_h \otimes a) = V^*\big(T_h \otimes \rho(a)\big)V, \quad h \in L^2(X\times X),\ a \in A.$$

It follows by linearity that

$$S_\varphi(T_k) = V^* T_{\rho\circ k} V, \quad k \in L^2(X\times X) \odot A.$$

Now choose arbitrary $k \in L^2(X\times X, A)$. By Proposition 1.79 there exists a sequence $(k_i)_{i\in\mathbb{N}} \subseteq L^2(X\times X) \odot A$ with $\|k_i - k\|_2 \to 0$. Using (2.5), (2.20), Lemma 2.1, and the fact that φ is bounded, we obtain

$$\begin{aligned}
S_\varphi(T_k) = \lim_{i\to\infty} S_\varphi(T_{k_i}) &= \lim_{i\to\infty} V^* T_{\rho\circ k_i} V \\
&= V^*\big(\lim_{i\to\infty} T_{\rho\circ k_i}\big)V \\
&= V^*\big(\lim_{i\to\infty} \tilde\rho(T_{k_i})\big)V \\
&= V^*\tilde\rho(T_k)V.
\end{aligned}$$

Thus the map $T \mapsto V^*\tilde\rho(T)V$ is a completely positive extension of S_φ to $\mathcal{K}(L^2(X)) \otimes_{\min} A$. Since $\mathrm{id} \otimes \rho$ is a representation of $\mathcal{B}(L^2(X)) \otimes_{\min} A$ on $L^2(X) \otimes \mathcal{H}_\rho$ it follows that the map $T \mapsto V^*\tilde\rho(T)V$ is an extension of S_φ to a completely positive map on

$\mathcal{B}(L^2(X)) \otimes_{\min} A$.

The norm equality holds because Theorem 2.9 applies if either condition here is satisfied.

□

We have achieved a step in the direction of goal (iii). Further investigation in terms of the space $\mathcal{S}_2(X, X; A)$ will be conducted when we specialise to the discrete case.

Let us now prove a transference result for positive multipliers, linking complete positivity of S_F and $S_{\mathcal{N}(F)}$.

Proposition 4.7. *Let (A, G, α) be a C^*-dynamical system and $F : G \to \mathcal{CB}(A)$ a bounded function. The following are equivalent:*

i. S_F is completely positive as a map on $A \rtimes_{\alpha,r} G$;

ii. $S_{\mathcal{N}(F)}$ is completely positive as a map on $\mathcal{K}(L^2(G)) \otimes_{\min} A$.

Moreover, if the conditions hold then $\|S_F\|_{\mathrm{cb}} = \|S_{\mathcal{N}(F)}\|_{\mathrm{cb}}$ and $S_{\mathcal{N}(F)}$ has a completely positive extension to $\mathcal{B}(L^2(G)) \otimes_{\min} A$. Finally, if A is unital and G is discrete then $\|F(e)\|_{\mathrm{cb}} = \|S_F\|_{\mathrm{cb}}$.

Proof. We follow the proof of Theorem 3.11, using complete positivity in place of complete boundedness.

(i)⇒(ii) Suppose that S_F is completely positive, so that in particular F is a Herz–Schur (A, G, α)-multiplier. Let $(\theta, \mathcal{H}_\theta)$ be a faithful representation of A on a separable Hilbert space. By Stinespring's Theorem, Theorem 1.23, there exist a representation $(\rho, \mathcal{H}_\rho)$ of $A \rtimes_{\alpha,\theta} G$, and an operator $V \in \mathcal{B}(L^2(G, \mathcal{H}_\theta), \mathcal{H}_\rho)$, such that

$$S_F(T) = V^*\rho(T)V, \quad T \in A \rtimes_{\alpha,\theta} G. \tag{4.17}$$

Let $q : A \rtimes_\alpha G \to A \rtimes_{\alpha,\theta} G$ denote the canonical quotient map and $\tilde{\rho} := \rho \circ q$, which is a non-degenerate representation of $A \rtimes_\alpha G$. By Proposition 1.91 there exist a representation $\rho_A : A \to \mathcal{B}(\mathcal{H}_\rho)$ and a strongly continuous unitary representation $\rho_G : G \to \mathcal{B}(\mathcal{H}_\rho)$

such that $\tilde{\rho} = \rho_A \rtimes \rho_G$. Let $f \in L^1(G, A)$. Using (4.17) with $T = \pi^\theta \rtimes \lambda^\theta(f)$ we have

$$\int_G \pi^\theta\Big(F(s)\big(f(s)\big)\Big)\lambda_s^\theta \, ds = V^* \left(\int_G \rho_A\big(f(s)\big)\rho_G(s)\, ds\right) V. \tag{4.18}$$

Let $a \in A$ and $(f_U)_U$ be a Dirac family at $t \in G$. Taking $f = f_U \otimes a$ in (4.18), using equation (3.11) and Lemma 3.10, we obtain a null set $M \subseteq G$ such that

$$\pi^\theta\big(F(t)(a)\big)\lambda_t^\theta = V^* \rho_A(a)\rho_G(t)V, \quad a \in A,\ t \in G \setminus M. \tag{4.19}$$

For each $s \in G$ define

$$\mathcal{V} : G \to \mathcal{B}(L^2(G, \mathcal{H}_\theta), \mathcal{H}_\rho);\ \mathcal{V}(s) := \rho_G(s^{-1})V\lambda_s^\theta, \quad s \in G.$$

As in the proof of Theorem 3.11 we have $\mathcal{V} \in L^\infty(G, \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho))$. By (4.19) we have, for any $(s,t) \in G \times G$ such that $ts^{-1} \in G \setminus M$, and any $a \in A$,

$$\begin{aligned}
\mathcal{V}(t)^* \rho_A(a)\mathcal{V}(s) &= \lambda_{t^{-1}}^\theta V^* \rho_G(t)\rho_A(a)\rho_G(s^{-1})V\lambda_s^\theta \\
&= \lambda_{t^{-1}}^\theta V^* \rho_A\big(\alpha_t(a)\big)\rho_G(ts^{-1})V\lambda_s^\theta \\
&= \lambda_{t^{-1}}^\theta \pi^\theta\Big(F(t)\big(\alpha_t(a)\big)\Big)\lambda_{ts^{-1}}^\theta \lambda_s^\theta \\
&= \pi^\theta\Big(\alpha_{t^{-1}}\big(F(t)\big(\alpha_t(a)\big)\big)\Big)\lambda_t^\theta \\
&= \pi^\theta\big(\mathcal{N}(F)(s,t)(a)\big).
\end{aligned}$$

Since the set $\{(s,t) : ts^{-1} \in M\}$ is null under the product measure Theorem 4.6 implies that $\mathcal{N}(F)^{\pi^\theta}$ is a Schur $\pi^\theta(A)$-multiplier with $S_{\mathcal{N}(F)^{\pi^\theta}}$ completely positive. As in (2.6) we have

$$S_{\mathcal{N}(F)^{\pi^\theta}} \circ (\mathrm{id} \otimes \pi^\theta) = (\mathrm{id} \otimes \pi^\theta) \circ S_{\mathcal{N}(F)};$$

it follows that $S_{\mathcal{N}(F)}$ is also completely positive.
(ii)$\Rightarrow$(i) The proof of Theorem 3.11 shows that S_F is the restriction of $S_{\mathcal{N}(F)^\theta}$ to $A \rtimes_{\alpha,\theta} G$; thus complete positivity of S_F follows immediately from the complete positivity of $S_{\mathcal{N}(F)}$ and equation (2.6).

The first norm equality holds because Theorem 3.11 applies if either condition holds. For the second we apply [43, Proposition 3.6]: suppose A is unital with identity 1_A;

since G is discrete $\pi(1_A)$ is the identity element of $A \rtimes_{\alpha,r} G$ and so

$$\|S_F\|_{\mathrm{cb}} = \|S_F\big(\pi(1_A)\big)\| = \|\pi\big(F(e)(1_A)\big)\| = \|F(e)(1_A)\| = \|F(e)\|_{\mathrm{cb}}$$

as $F(e)$ is clearly completely positive.

Finally, the statement about extending $S_{\mathcal{N}(F)}$ to a completely positive map acting on $\mathcal{B}(L^2(G)) \otimes_{\min} A$ follows exactly as in the proof of Theorem 4.6. □

4.3.2 Positivity for multipliers on discrete spaces

We now specialise to the discrete case. In this section all measure spaces will be discrete spaces with counting measure, and integrals will be written as sums.

Continuing the investigation of positive Schur and Herz–Schur multipliers, our first goal is to link the results so far with positivity of the operators $T_k \in \mathcal{S}_2(X, X; A)$, which are elements of $\mathcal{B}(\ell^2(X, \mathcal{H}))$. The following definition reduces to the usual one, given by *e.g.* Bekka–Harpe–Valette [7, Definition C.1.1], when $A = \mathbb{C}$.

Definition 4.8. Let X be a set equipped with counting measure, A a C^*-algebra, and $k \in \ell^2(X \times X, A)$. Then k is called *hermitian* if $k(x,y)^* = k(y,x)$ for all $x, y \in X$. A hermitian kernel k is called *positive-definite* if, for any $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in X$, the matrix

$$\big(k(x_i, x_j)\big)_{i,j=1}^n \in M_n(A)$$

is positive.

Proposition 4.9. *Let X be a set equipped with counting measure, $A \subseteq \mathcal{B}(\mathcal{H})$ a C^*-algebra, and $k \in \ell^2(X \times X, A)$ a hermitian function. The following are equivalent:*

i. k is positive-definite;

ii. T_k is a positive element of $\mathcal{B}(\ell^2(X, \mathcal{H}))$.

Proof. Suppose that k is positive-definite and $\xi \in \ell^2(X, \mathcal{H})$ is compactly (*i.e.* finitely) supported, say $\operatorname{supp}\xi = \{x_1, \dots, x_n\}$. Then

$$\begin{aligned}\langle T_k\xi, \xi\rangle = \sum_{x\in X} \langle T_k\xi(x), \xi(x)\rangle &= \sum_{x\in X} \left\langle \sum_{y\in X} k(x,y)\big(\xi(y)\big), \xi(x)\right\rangle \\ &= \sum_{i,j=1}^{n} \big\langle k(x_i,x_j)\big(\xi(x_j)\big), \xi(x_i)\big\rangle \geq 0,\end{aligned}$$

since k is positive-definite. Now if $\eta \in \ell^2(X, \mathcal{H})$ is arbitrary there exists, by Proposition 1.79, a sequence $(\eta_i)_{i\in\mathbb{N}}$ of compactly supported functions in $\ell^2(X, \mathcal{H})$ converging to η; thus $\langle T_k\eta_i, \eta_i\rangle \to \langle T_k\eta, \eta\rangle$. It follows from the above display that the last is a limit of non-negative numbers, so is non-negative.

Now suppose k is not positive-definite, so there exist $\xi_1, \dots, \xi_n \in \mathcal{H}$ and $x_1, \dots, x_n \in X$ such that

$$\sum_{i,j=1}^{n} \langle k(x_i, x_j)\xi_j, \xi_i\rangle < 0.$$

Define $\xi \in \ell^2(X, \mathcal{H})$ by

$$\xi(x) := \begin{cases} \xi_i, & \text{if } x = x_i \\ 0, & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}\langle T_k\xi, \xi\rangle = \sum_{x\in X} \langle T_k\xi(x), \xi(x)\rangle &= \sum_{x\in X} \left\langle \sum_{y\in X} k(x,y)\big(\xi(y)\big), \xi(x)\right\rangle \\ &= \sum_{i,j=1}^{n} \langle k(x_i,x_j)\xi_j, \xi_i\rangle < 0,\end{aligned}$$

so T_k is not a positive operator. □

Definition 4.10. Let X be a set equipped with counting measure, $A \subseteq \mathcal{B}(\mathcal{H})$ a C^*-algebra, and $\varphi : X \times X \to \mathcal{CB}(A, \mathcal{B}(\mathcal{H}))$ a function. We say φ is *of positive type* if, for any $n \in \mathbb{N}$, any $x_1, \dots, x_n \in X$, and any positive matrix $(a_{i,j})_{i,j=1}^n \in M_n(A)$ the matrix

$$\big(\varphi(x_j, x_i)(a_{i,j})\big)_{i,j=1}^n \in M_n(A)$$

is positive.

Observe that if $k \in \ell^2(X \times X, A)$ is positive-definite and φ is of positive type then $\varphi \cdot k \in \ell^2(X \times X, A)$ is positive-definite. Indeed, take $x_1, \ldots, x_n \in X$. Then

$$\big(\varphi \cdot k(x_i, x_j)\big)_{i,j=1}^n = \Big(\varphi(x_j, x_i)\big(k(x_i, x_j)\big)\Big)_{i,j=1}^n.$$

Since k is positive-definite the matrix $(k(x_i, x_j))_{i,j=1}^n$ is a positive element of $M_n(A)$, so that positivity of the displayed matrix is immediate from the fact that φ is of positive type.

Remark 4.11. Suppose that $\varphi : X \times X \to \mathcal{CB}(A, \mathcal{B}(\mathcal{H}))$ is of positive type. Then φ automatically satisfies the following (apparently stronger) condition: for any $m, n \in \mathbb{N}$, any $x_1, \ldots, x_m \in X$, and any collection of matrices $C_{p,q} \in M_n(A)$ such that $(C_{p,q})_{p,q=1}^m \in M_{mn}(A)^+$, the matrix

$$\big(\varphi(x_q, x_p)^{(n)}(C_{p,q})\big)_{p,q=1}^m \in M_{mn}(\mathcal{B}(\mathcal{H}))$$

is positive.

Proof. Suppose that φ is of positive type, and take $C_{p,q} \in M_n(A)$ such that $(C_{p,q})_{p,q=1}^m \in M_{mn}(A)^+$ and $x_1, \ldots, x_m \in X$. Let $C_{p,q} = (a_{i,j,p,q})_{i,j=1}^n$ and define $b_{k,l}$, $1 \le k, l \le mn$, by

$$b_{k,l} := a_{k',l',p,q} \quad \text{when } k = (p-1)m + k',\ l = (q-1)m + l',$$

$1 \le p, q \le n,\ 1 \le k', l' \le n$, so that $(C_{p,q})_{p,q=1}^m = (b_{k,l})_{k,l=1}^{mn} \in M_{mn}(A)^+$. Then we have

$$\begin{aligned}\big(\varphi(x_q, x_p)^{(n)}(C_{p,q})\big)_{p,q=1}^m &= \big(\varphi(x_q, x_p)(a_{i,j,p,q})\big)_{i,j,p,q}\\ &= \big(\varphi(y_k, y_l)(b_{k,l})\big)_{k,l=1}^{mn},\end{aligned}$$

where $y_{(s-1)m+r} = x_s$. So if φ is of positive type then, since $(b_{k,l})_{k,l=1}^{mn}$ is a positive matrix, it follows that φ automatically satisfies the condition given above. $\square$

Proposition 4.12. *Let X be a set equipped with counting measure, $A \subseteq \mathcal{B}(\mathcal{H})$ a C^*-algebra, and $\varphi : X \times X \to \mathcal{CB}(A, \mathcal{B}(\mathcal{H}))$ a bounded function. The following are equivalent:*

i. φ is of positive type;

ii. S_φ is completely positive on $\mathcal{K}(\ell^2(X)) \otimes_{\min} A$.

In particular, if φ is of positive type then it is automatically a Schur A-multiplier. If the conditions hold then S_φ is completely positive on $\mathcal{B}(\ell^2(X)) \otimes_{\min} A$. Moreover, if the conditions hold and A is unital then $\|\varphi\|_{\mathfrak{S}} = \sup_{x \in X} \|\varphi(x,x)\|_{\mathrm{cb}}$.

Proof. (i)⇒(ii) Let $(T_{k_{i,j}})_{i,j=1}^n$ be a positive matrix, acting on $\ell^2(X, \mathcal{H})^n$, with entries in $\mathcal{S}_2(X, X; A)$. As in Proposition 4.9 it suffices to show that $\left\langle S_\varphi^{(n)}(T_{k_{i,j}})\xi, \xi \right\rangle \geq 0$ for finitely supported $\xi \in \ell^2(X, \mathcal{H})^n$. Suppose $\xi = (\xi_i)_{i=1}^n \in \ell^2(X, \mathcal{H})^n$ has finite support. Then

$$\begin{aligned}
\left\langle S_\varphi^{(n)}(T_{k_{i,j}})\xi, \xi \right\rangle &= \sum_{i,j=1}^n \left\langle T_{\varphi \cdot k_{i,j}} \xi_j, \xi_i \right\rangle \\
&= \sum_{x \in \operatorname{supp} \xi} \sum_{i,j=1}^n \left\langle T_{\varphi \cdot k_{i,j}} \xi_j(x), \xi_i(x) \right\rangle \\
&= \sum_{x,y \in \operatorname{supp} \xi} \sum_{i,j=1}^n \left\langle \varphi \cdot k_{i,j}(x,y)\big(\xi_j(y)\big), \xi_i(x) \right\rangle \\
&= \sum_{x,y \in \operatorname{supp} \xi} \sum_{i,j=1}^n \left\langle \varphi(y,x)\big(k_{i,j}(x,y)\big)\big(\xi_j(y)\big), \xi_i(x) \right\rangle \geq 0,
\end{aligned}$$

because if $\operatorname{supp} \xi = \{x_1, \ldots, x_m\}$ then the matrix $(k_{i,j}(x_p, x_q))$ is positive in $M_{mn}(A)$, by Proposition 4.9, so the result follows since φ is of positive type. This shows that S_φ is completely positive on $\mathcal{S}_2(X, X; A)$, and therefore on $\mathcal{K}(\ell^2(X)) \otimes_{\min} A$.

(ii)⇒(i) By Theorem 4.6 $\varphi(x,y)(a) = V(y)^* \rho(a) V(x)$ $(a \in A,\ x, y \in X)$. Given $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, and a positive matrix $(a_{i,j})_{i,j=1}^n \in M_n(A)$, let $\hat{V}$ denote the $n \times n$ matrix with diagonal entries $\hat{V}_{i,i} = V(x_i)$ and all other entries 0. Since ρ is completely positive the matrix

$$\hat{V}^* \big(\rho(a_{i,j})\big)_{i,j=1}^n \hat{V}$$

is a positive element of $M_n(A)$, and we have

$$\hat{V}^*\big(\rho(a_{i,j})\big)_{i,j=1}^n \hat{V} = \big(V(x_i)^*\rho(a_{i,j})V(x_j)\big)_{i,j=1}^n = \big(\varphi(x_j, x_i)(a_{i,j})\big)_{i,j=1}^n,$$

so φ is of positive type.

The statement about extending S_φ to $\mathcal{B}(\ell^2(X)) \otimes_{\min} A$ follows from Theorem 4.6. For the norm equality we calculate $\|S_\varphi\|_{\mathrm{cb}}$. Assume A is unital with identity 1_A. First observe that if the conditions hold then $\varphi(x, x)$ is completely positive, since condition (ii) of Theorem 4.6 holds. In particular, for any $x \in X$, by Paulsen [43, Proposition 3.6],

$$\|\varphi(x, x)\|_{\mathrm{cb}} = \|\varphi(x, x)(1_A)\| = \|S_\varphi(\chi_{\{x\}\times\{x\}} \otimes 1_A)\|,$$

so that $\|\varphi(x, x)\|_{\mathrm{cb}} \leq \|S_\varphi\|_{\mathrm{cb}}$. To prove the reverse inequality let $E \subseteq X$ be a finite set. View $\ell^2(E)\otimes\mathcal{H}$ as a subspace of $\ell^2(X)\otimes\mathcal{H}$, and $M_{|E|}\otimes A = \mathcal{S}_2(E, E; A)$ as a subspace of $\mathcal{S}_2(X, X; A)$. Clearly the map S_φ leaves $\mathcal{S}_2(E, E; A)$ invariant, and the proof of (i)$\Rightarrow$(ii) shows that the restriction $S_{\varphi,E}$ of S_φ to $\mathcal{S}_2(E, E; A)$ is completely positive. Since $\mathcal{S}_2(E, E; A)$ is a unital C^*-algebra, with identity denoted by I, applying [43, Proposition 3.6] again we obtain

$$\|S_{\varphi,E}\|_{\mathrm{cb}} = \|S_{\varphi,E}(I)\| = \max_{x\in E} \|\varphi(x, x)(1_A)\| = \max_{x\in E} \|\varphi(x, x)\|_{\mathrm{cb}} \leq \sup_{x\in X} \|\varphi(x, x)\|_{\mathrm{cb}}. \tag{4.20}$$

Since the spaces $(\mathcal{S}_2(E, E; A))_{E\subseteq X}$ form an upwards directed net, ordered by inclusion of finite subsets of X, which is dense in $\mathcal{S}_2(X, X; A)$, it follows from (4.20) that $\|S_\varphi\|_{\mathrm{cb}} \leq \sup_{x\in X} \|\varphi(x, x)\|_{\mathrm{cb}}$. □

Let us now consider Herz–Schur multipliers of a C^*-dynamical system (A, G, α). In keeping with our assumption of discrete measure spaces we consider only groups with the discrete topology; since we have a standing assumption of second-countability this means that we consider a C^*-dynamical system (A, G, α) where G is a countable discrete group.

Definition 4.13. Let (A, G, α) be a C^*-dynamical system with G a discrete group, and $F : G \to \mathcal{CB}(A)$ a bounded function. Define F to be *of positive type* when $\mathcal{N}(F)$ is of positive type; that is, F is of positive type if, for any $n \in \mathbb{N}$, any $s_1, \dots s_n \in G$, and any positive matrix $(a_{i,j}) \in M_n(A)$, the matrix

$$\left(\alpha_{s_i^{-1}}\left(F(s_i s_j^{-1})\big(\alpha_{s_i}(a_{i,j})\big)\right)\right)_{i,j=1}^{n}$$

is a positive element of $M_n(A)$.

Observe that if $A = \mathbb{C}$ the above definition reduces to Definition 1.48, by Paulsen [43, Theorem 3.7].

The following summarises our results on Herz–Schur multipliers of a C^*-dynamical system.

Corollary 4.14. *Let (A, G, α) be a C^*-dynamical system with G a countable discrete group, and $F : G \to \mathcal{CB}(A)$ a bounded function. The following are equivalent:*

i. F is of positive type;

ii. $S_{\mathcal{N}(F)}$ is completely positive;

iii. S_F is completely positive.

In particular, if F is of positive type then F is automatically a Herz–Schur multiplier of (A, G, α). Moreover, if A is unital then $\|F(e)\|_{\mathrm{cb}} = \|S_{\mathcal{N}(F)}\|_{\mathrm{cb}} = \|S_F\|_{\mathrm{cb}} = \|F\|_{\mathrm{HS}}$.

Proof. The equivalence of (i) and (ii) follows from Definition 4.13 and Proposition 4.12. That (ii) and (iii) are equivalent is Proposition 4.7. The norm equality also follows from the quoted results. □

4.3.3 Connections with other positive multipliers

The multipliers considered in Section 3.3 have a notion of positivity, which we now compare with the one arrived at here. The *centre* of the C^*-algebra A will be denoted $\mathcal{Z}(A)$.

Definition 4.15. Let (A, G, α) be a C^*-dynamical system with G a discrete group.

- A function $T : G \times A \to A$, which is linear in the second variable, is said to be *positive-definite in the sense of Bédos–Conti*, or *BC positive-definite*, if for any $n \in \mathbb{N}$, any $s_1, \ldots, s_n \in G$, and any $a_1, \ldots, a_n \in A$, the matrix

$$\Big(\alpha_{s_i}\big(T(s_i^{-1}s_j, \alpha_{s_i^{-1}}(a_i^* a_j))\big)\Big)_{i,j=1}^n$$

  is a positive element of $M_n(A)$. This definition was given by Bédos and Conti [5, Definition 4.7] in the more general case of a twisted C^*-dynamical system; here we consider only the trivial twist and have simplified the definition accordingly.

- A function $h : G \to \mathcal{Z}(A)$ is said to be *positive-definite in the sense of Dong–Ruan*, or *DR positive-definite*, if for any $n \in \mathbb{N}$ and any $s_1, \ldots, s_n \in G$ the matrix

$$\Big(\alpha_{s_j}\big(h(s_i^{-1}s_j)\big)\Big)_{i,j=1}^n$$

  is a positive element of $M_n(A)$. This definition was given by Dong and Ruan [20, page 436]; only centre-valued functions are considered because this is a necessary condition for such a map to be a multiplier of the reduced crossed product in their sense.

- A function $\phi : G \to A$ is said to be *α-positive-definite* if for any $n \in \mathbb{N}$ and any $s_1, \ldots, s_n \in G$ the matrix

$$\Big(\alpha_{s_i}\big(\phi(s_i^{-1}s_j)\big)\Big)_{i,j=1}^n$$

  is a positive element of $M_n(A)$. This definition was given by Anantharaman-Delaroche [1, Définition 2.1], and used by Bédos–Conti [6, page 3]; we will only consider this definition when ϕ takes values in $\mathcal{Z}(A)$, since that is the case considered by the authors above (see [1, Théorème 3.3] and [6, Section 2]).

We now investigate how the above notions compare to Definition 4.13; I will not consider DR positive-definiteness since it is similar to the third notion — the difference in indexing is not crucial.

Recall from Section 3.3 that a function $T : G \times A \to A$, which is linear in the second variable, is a multiplier of (A, G, α) in the sense of Bédos–Conti if and only if the function

$$F_T : G \to \mathcal{CB}(A);\ F_T(t)(a) := T(t, a), \quad t \in G,\ a \in A,$$

is a Herz–Schur (A, G, α)-multiplier. Let $n \in \mathbb{N}$, $s_1, \dots, s_n \in G$, and $a_1, \dots, a_n \in A$; the calculation

$$\begin{aligned}\Big(\alpha_{s_i}\big(T(s_i^{-1}s_j, \alpha_{s_i^{-1}}(a_i^* a_j))\big)\Big)_{i,j=1}^n &= \Big(\alpha_{s_i}\big(F_T(s_i^{-1}s_j)\big(\alpha_{s_i^{-1}}(a_i^* a_j)\big)\big)\Big)_{i,j=1}^n \\ &= \big(\mathcal{N}(F_T)(s_j^{-1}, s_i^{-1})(a_i^* a_j)\big)_{i,j=1}^n\end{aligned}$$

implies that T is BC positive-definite if and only if F_T is a Herz–Schur (A, G, α)-multiplier of positive type, since any positive matrix in $M_n(A)$ is a sum of matrices of the form $(a_i^* a_j)_{i,j=1}^n$ by Takesaki [55, Lemma IV.3.1].

Now suppose that $\phi : G \to \mathcal{Z}(A)$ is α-positive-definite. To ϕ we associate the function

$$F_\phi : G \to \mathcal{CB}(A);\ F_\phi(t)(a) := \phi(t)a, \quad t \in G,\ a \in A.$$

Let $n \in \mathbb{N}$, $s_1, \dots, s_n \in G$, and $(a_{i,j})_{i,j=1}^n$ a positive matrix in $M_n(A)$; then

$$\begin{aligned}\big(\mathcal{N}(F_\phi)(s_j^{-1}, s_i^{-1})(a_{i,j})\big)_{i,j=1}^n &= \Big(\alpha_{s_i}\big(F_\phi(s_i^{-1}s_j)\big(\alpha_{s_i^{-1}}(a_{i,j})\big)\big)\Big)_{i,j=1}^n \\ &= \Big(\alpha_{s_i}\big(\phi(s_i^{-1}s_j)\big)a_{i,j}\Big)_{i,j=1}^n,\end{aligned}$$

which is positive as the Schur product of a positive matrix in $M_n(\mathcal{Z}(A))$ and a positive matrix in $M_n(A)$ by [6, Lemma 2.1] (see also [41, Lemma 3.1]). Conversely, if $\mathcal{N}(F_\phi)$ is a Herz–Schur (A, G, α)-multiplier of positive type then, since the matrix in $M_n(A)$ with all entries equal to 1_A is positive, we have for any $n \in \mathbb{N}$ and any $s_1, \dots, s_n \in G$

$$0 \leq \Big(\mathcal{N}(F_\phi)(s_j^{-1}, s_i^{-1})(1_A)\Big)_{i,j=1}^n = \Big(\alpha_{s_i}\big(\phi(s_i^{-1}s_j)\big)\Big)_{i,j=1}^n,$$

which shows that ϕ is an α-positive function.

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